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THE MARINER'S MIRROR,

PART III.

Being a New and Compendious

SYSTEM OF LOGARITHMS

In all the different KINDS, viz.

I. *Nautical* LOGARITHMS invented by Mr. WRIGHT.
II. *Natural* LOGARITHMS by Lord NEPER.
III. *Common* LOGARITHMS by Mr. BRIGGS.

WITH

Their Application in the Operations of ARITHMETIC, the DOCTRINE of RATIOS, Natural PHILOSOPHY, the *Coteſian* GEOMETRY; and NAVIGATION, in Particular.

The Whole illuſtrated by the *Logiſtic* CURVE at large; with the CONSTRUCTION and DELINEATION of all the *Logarithmic* LINES and SCALES.

BY BENJAMIN MARTIN.

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PREFACE.

AS my TREATISE *of* LOGARITHMS *has been long out of Print, and a Demand for it ſtill continuing, I have been induced to epitomiſe this admirable Science, and have given it quite a new Dreſs, by adding ſeveral important Articles hitherto wanting in Books on this Subject, eſpecially reſpecting the* INVENTION *and* Application *of theſe artificial Numbers to* Nautical USES.

The Invention *of* LOGARITHMS *has been unjuſtly* 1614 *aſcribed to Lord* NEPER, *a* Scotch *Baron, whereas it is in Reality due only to our Countryman Mr.* EDWARD WRIGHT, *as I have fully demonſtrated.* 1599 *Mr.* WRIGHT*'s Table of* LATITUDES *was the firſt* SYSTEM *of* LOGARITHMS *the World ever ſaw, and are peculiarly adapted to the Service of* NAVIGATION; *whereas the Logarithms afterwards publiſhed by Lord* Neper, *were of ſuch a Nature, and Form, as rendered them of little Uſe, till they received a Transformation by Mr.* HENRY BRIGGS, *who might juſt as well have made the* Trigonometrical CANON *of* LOGARITHMS *in preſent Uſe from Mr.*

Wright*'s, as from Lord* Neper's Form *and* Module; *if he had but conſidered it as thoroughly as the perſpicacious Dr.* HALLEY *did.*

As Logarithms are artificial Numbers, or rather natural Numbers artificially applied, their origin from a Series of Numbers in arithmetical and geometrical Progreſſion *is much more naturally and properly deduced than from* hyperbolic Spaces; *and being expreſſed or repreſented by* Lines, *gives a much eaſier and clearer Idea of them than can be had from the intricate Theory of a* Conic Section.

The Logiſtic CURVE *is here drawn ſo large, that the Nature and Conſtruction of all* Logarithmic LINES *and* SCALES *may be eaſily ſeen to originate from it; of which no leſs than* Eight, *are here appoſitely placed to correſpond to the ſaid Curve, are very large and properly divided, and receive from hence the moſt natural Elucidation of their Theory.*

The Application of Logarithms to NAVIGATION *is with the greateſt Propriety; for this Science is perhaps the only one that is wholly of a* Logiſtic *Nature; the Way of the Ship upon the Sea is upon an equiangular Curve or Rhumb; this very Curve is projected* in Plano, *into the* Logarithmic Spiral, *and therefore no wonder if all* nautical *Computations are founded in, and derived from a* Logiſtical Meridian Line, *and* Logarithmic Tangents. *A Treatiſe of Logarithms therefore is an eſſential Part of a Treatiſe on Navigation, and ought to be regarded as the* Mariner's Premier; *to evince this more fully, it is here ſhewn how all the*

the Caſes of Mercator's *ſailing are ſolved by the common Canon of Logarithm Tangents; which are the ſame Thing as* Meridional Parts, *but in another* SCALE *and* FORM.

Their Application to the Common Rules of ARITHMETIC, *and the Rationale of each Operation, I have endeavoured to make as general, eaſy, and apparent as poſſible.*

The Uſefulneſs, or rather Neceſſity, of Logarithms in Natural PHILOSOPHY *is ſufficiently evinced by two capital Examples. And the* Doctrine, Notation, *and* Arithmetic *of Ratios, ſo little underſtood, is here placed in a New, and, it is hoped, a very advantageous Light.*

As the Coteſian Matheſis *is a Sort of* Logiſtical Geometry, *and is, for the moſt Part, locked up from the Public in the* Harmonia Menſurarum *(a Book out of Print, and rarely to be met with) I thought I could not do a more acceptable Service than to explain its Principles here, and apply them in a few important Examples to practice. In ſhort, the preſent Tract is but, as it were, An* INTRODUCTION *to that admirable Work of Mr.* COTES.

As this Compendium *makes the* Third Part *of the* MARINER'S MIRROR, *ſo a New* TREEATISE *of* TRIGONOMETRY *is now ready for the Preſs, and will ſoon appear in the* Fourth Part; *wherein the fundamental Propoſitions of* Plain *and* Spherical TRIANGLES *will be demonſtrated in a Method entirely new, and much more compendious and eaſy than is to be found in any other Book on the Subject.*

As to TABLES *of* LOGARITHMS, *theſe I have for many Reaſons, purpoſely omitted; for* (1) *the Trouble of correcting ſuch a Set of Tables would be very great, and which, at my Time of Life, I could not undertake.* (2.) *It would inhance the Price of the Book without any great Occaſion, becauſe* (3) *there are ſuch Tables in almoſt every Book of* Navigation, *and practical Mathematics.* (4.) *But the principal Reaſon is, that where an Exactneſs is not required beyond all kind of Uſe, the* GUNTER *is found ſufficient, which is only ſuch a Table of Logarithms protracted on a Line; and therefore I ſhall take the Liberty of recommending to the Reader the Uſe of my new improved and univerſal* SLIDING-RULE, *with it's Application in* 60 *Practical Problems in all Parts of* MATHEMATICS *and* PHILOSOPHY.

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A New

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A New and Compendious

SYSTEM

OF

LOGARITHMS.

CHAP. I.

Of the NATURE *and* GENESIS *of* LOGARITHMS; *and their* USE *in facilitating all the* OPERATIONS *of Vulgar* ARITHMETIC.

1. IF any Number or Quantity (a) be multiplied by another Number (e), the Product is ae; this again multiplied by (e), produces aee; this again multiplied by (e), is $aeee$, and ſo on continually. The ſeveral Terms or Products ariſing by ſuch a conſtant Multiplication, will ſtand as below:

$a.\ ae.\ aee.\ aeee.\ aeeee.\ aeeeee.$ &c. Or thus,
$a.\ ae^1.\ ae^2.\ ae^3.\ ae^4.\ ae^5.\ ae^6.\ -\ -\ -\ ae^n.$

2. The Terms of ſuch a Series are in *Geometrical Progreſſion*; and the Indices or Numbers expounding the Powers of (e) in each Term, viz. 0. 1. 2. 3. 4. 5. 6. - - - - n, are Numbers in *Arithmetical Progreſſion*, having all *one common Difference*, which in this Series is unity, or 1.

3. If we put $a=1$, the Series will be more ſimple, viz. 1. e^{1}. e^{2}. e^{3}. e^{4}, e^{5}. e^{6}. - - - e^{n}. this is a Series of increaſing Terms; and the Exponents 1. 2. 3. 4. 5. &c. are *poſitive*. But there may be the like Series of decreaſing Terms by Diviſion, viz. $\frac{1}{e.}$ $\frac{1}{ee.}$ $\frac{1}{eee.}$ $\frac{1}{eeee.}$ &c. which may be thus expreſſed with *Negative Indices*, viz. 1. $\overline{e.}^{1}$ $\overline{e.}^{2}$ $\overline{e.}^{3}$ $\overline{e.}^{4}$ - - - $\overline{e.}^{n}$ So that the Series on each Side Unity, will properly ſtand thus, $\overline{e.}^{n}$ - - - $\overline{e.}^{4}$ $\overline{e.}^{3}$ $\overline{e.}^{2}$ $\overline{e.}^{1}$ 1. e^{1}. e^{2}. e^{3}. e^{4}. - - - e^{n}; the Terms being continued each Way to the Number n. Note, $e^{\circ}=1$, becauſe $\frac{1}{e}\times e=\frac{e}{e}=1$, or e°.

4. Sometimes the conſtant Multiplier is a *Binomial*, and conſiſts of the Sum or Difference of two Quantities, as $a+x$, $a-x$; or $1+x$, $1-x$. And then the Terms of the Series will ſtand thus, 1. $\overline{1+x}^{1}$ · $\overline{1+x}^{2}$ · $\overline{1+x}^{3}$. $\overline{1+x}^{4}$ · - - - $\overline{1+x}^{n}$ · or for the decreaſing Series the Terms will be thus expreſſed, 1. $\overline{1-x}^{1}$ · $\overline{1-x}^{2}$ · $\overline{1-x}^{3}$ · $\overline{1-x}^{4}$ · - - - $\overline{1-x}^{n}$.

5. This Series of *Geometrical Terms* may be expreſſed in pure Numbers, with the *Arithmetical Series of Exponents*, thus:

Arith. 0. 1. 2. 3. 4. 5. 6. &c.
Geomet. 1. 3. 9. 27. 81. 243. 729. &c.

6. Theſe arithmetical *Exponents* are called (in *Greek*) LOGARITHMS, becauſe they expound or expreſs the Place

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Place which every Term in the Geometric Series holds from Unity, or from each other, which is called the *Ratio* of thoſe Terms to Unity, or to each other. Thus 2 is the *Exponent* or *Logarithm* of 9 in the foregoing Geometrical Series, or it ſhews that 9 is the 2d Term from Unity, and that the *Ratio* (or Multiplication by 3) has been twice repeated to produce it. In like Manner, 5 is the *Logarithm* of 243, or of the Ratio of 243 to 1; and the Logarithm of Unity or 1 is 0, as that is not diſtant from itſelf.

7. From what has been premiſed, it is evident that ADDITION of Exponents, or Logarithms, anſwers to a MULTIPLICATION of the correſponding Numbers or Terms; thus $1+3=4$, and $3\times27=81$. Alſo $2+4=6$, and $9\times81=729$; and $0+5=5$, to which correſponds $1\times243=243$. And ſo in every other Caſe.

8. In like Manner a SUBTRACTION of Logarithms correſponds to a DIVISION of the reſpective Numbers; thus $4-1=3$, and $81\div3=27$. Alſo $6-2=4$, and $729\div9=81$; and ſo of others.

9. INVOLUTION of a Number to any Power, is correſpondent to multiplying the Logarithm thereof by 2, 3, 4, 5, *&c.* thus twice 2 is 4, and 81 is the *Square* of 9. Alſo 3 Times 2 is 6, and the *Cube* of 9 is 729; thus alſo, 4 Times 1 is 4, ſhewing 81 to be the *Biquatratic* Power of 3; and 5 Times 1 is 5, which ſhews 243 is the 5th or *Surſolid Power* of 3. And ſo on for other higher Powers.

10. EVOLUTION, or *extracting the Root* of any Number, is done by taking one *Half*, one *Third*, one *Fourth*, &c. of the Logarithm thereof; thus $\frac{1}{2}$ of 4 is 2, ſhewing the *Square Root* of 81 to be 9. Alſo $\frac{1}{3}$ of 6 is 2, which ſhews 9 to be the *Cube Root* of 729. Again $\frac{1}{4}$ of 4 is 1, and ſhews 3 is the *Biquadratic Root* of 81; and $\frac{1}{5}$ of 5 being 1, ſhews 3 is the *Surſolid Root* of 243; and ſo of others.

11. From what has been ſaid of this Series, it appears that all the Terms thereof are proportional to each other, or have the ſame Ratio in regard to magnitude; thus 9 is juſt as much bigger than 3, as 3 is bigger than 1; or $9:3::3:1$. Alſo 27 is in Magnitude to 9 as 3 is to 1; that is, $27:9::3:1$. Again, $81:27::27:9$; and $81:27::9:3$; and $729:243::81:27$. And ſo on throughout the Series; which is the Reaſon why they are called, *A Series of Numbers or Quantities in Geometrical Proportion.*

12. Hence the Practice by the RULE OF THREE of finding a *Fourth Proportional* to three given Numbers or Quantities; viz. the Product of the two Means is equal to that of the two Extremes; thus $81\times9=27\times27$ and $81\times3=27\times9$; and $729\times27=243\times81$; and ſo on. Therefore 3 Numbers being given to find the fourth, you divide the Product of the 2d and 3d by the firſt; and the Quotient is the Anſwer; thus, $\frac{27\times9}{81}=3$, or $\frac{27\times9}{3}=81$.

But this Operation is performed by only the *Addition and Subtraction* of the Logarithms of the given Numbers. Thus $3+2-4=1$, the Logarithm of 3. Or $3+2-1=4$, the Logarithm of 81. Again $5+4-3=6$ the Logarithm of 729 or $5+4-6=3$, the Logarithm of 27. Hence thoſe tedious Operations are greatly facilitated by theſe *Artificial Numbers or Logarithms.*

14. When any two Terms of the Series are compared, the Ratio is ſaid to be *ſimple*, as of 3 to 9, whoſe Logarithms are 1 and 2; but the Ratio of their *Squares* 9 and 81 is ſaid to be *duplicate*, becauſe their Logarithms 2 and 4 are double the former; and the Ratio of the Cubes of 3 and 9, viz. 27 and 729, is ſaid to be *triplicate*, as their Logarithms 3 and 6 are *triple* thoſe of the ſimple Ratio.

Ratio On the contrary, the Ratio of the *Square Roots, Cube Roots*, &c. is ſaid to be *ſubduplicate, ſubtriplicate*, &c. of the *Simple Ratio* of the Terms.

C H A P. II.

Of the NATURE *and* CONSTRUCTION *of the* Logiſtic *or* Logarithmetic CURVE; *of the univerſal* SERIES *or* SYSTEMS *of Logarithms thence deduced*; *and of the* Modular Ratio.

15. A Series of proportional Numbers or Quantities may be repreſented by right Lines, above or below that which denotes Unity. Thus let AB=1, and let CD, EF, GH, IK, *&c.* be Proportionals in the Scale above; and *cd, ef, gh, ik*, &c. in the Scale below Unity (Fig. 1.)

16. Then if theſe Quantities or Lines are placed upon a Right Line YZ of an indefinite Length, it is evident the proportional Lines AB, CD, EF, *&c.* will have their Logarithms expreſſed by 0, AC=1, AE=2, AG=3, AI=4, *&c.* as in the numerical Series at (5.) Alſo becauſe Logarithms are in *Arithmetical Progreſſion*, or *equidifferent*, therefore the proportional Terms AB, CD, EF, *&c.* will be *equidiſtant*, or the Intervals AC=CE=EG =GI, *&c.*

17. If the Tops of theſe Proportionals be connected by the Line WPX, that Line will be the *Logarithmetic* or *Logiſtic* CURVE; and the Line YZ is its Axis, or rather *Aſymptote*, becauſe the Curve towards W will conſtantly approach to, but can never coincide with it, as we ſhall ſee preſently.

18. Parallel to the Axis YZ draw B*m*, D*n*, F*o*, H*p*, &c. then will D*m*, F*n*, H*o*, K*p*, be the Differences of the Proportional Terms, and will be in the ſame Proportion

as the Terms themſelves. For ſince AB : CD : : CD : EF : : EF : GH, *&c.* we ſhall have (by Diviſion) AB−CD : AB : : CD−EF : CD : : EF−GH : EF, *&c.* That is, AB : CD : : Dm : Fn; and CD : EF : : Fn : Ho; &c. Therefore, $\frac{AB}{Dm}=\frac{CD}{Fn}=\frac{EF}{Ho}=\frac{GH}{Kp}$, *&c.* A given Quantity.

19. Let NS touch the Curve in N; and let NT, ML, be two other Terms in the ſame Series or Scale; and draw Nq parallel to the Axis YZ; then will the Triangles MNq and NTS be ſimilar; and therefore Mq : Nq : : NT : TS; therefore $\frac{NT \times Nq}{Mq}=TS$. But N$q$= TL=AC, a conſtant Quantity; alſo $\frac{NT}{Mq}$ is a conſtant Quantity (by 18) therefore the *Subtangent* TS is, in this Curve, a *conſtant Quantity*, or every where the ſame.

20. Let the Number TN$=x$, and its Logarithm AT$=g$, then when the Terms of the Series are indefinitely near, we have Nq=(TL=)$\dot{g}$, M$q=\dot{x}$, and ſo $\dot{x} : \dot{g} : : x :$ TS $=t$; whence $t\dot{x}=\dot{g}x$. Wherefore, by making the conſtant Quantity $t=1$, we have $\frac{\dot{x}}{x}=\dot{g}$, or *the Fluxion of the Logarithm of any Number is equal to the Fluxion of the Number divided by the Number itſelf.*

21. Hence the Meaſure of the Ratio of $x+\dot{x}$ to x or of LM to TN, is expreſſed in every different Curve by $t\times\frac{\dot{x}}{x}=\dot{g}$, therefore the *Logarithm of the ſame Ratio will be always as the Subtangent of the Curve*; becauſe, in this Caſe, $\frac{x}{\dot{x}}$ or $\frac{\dot{x}}{x}$ is a given Quantity (18.) On this Account the Subtangent (t) may be eſteemed the MODULE *of the Logarithms* in every ſuch Curve, or Scale of Proportionals belonging to it.

22. On the Point S, let there be erected the right Line SP, repreſenting another Proportional in the ſame Scale, then will the Ratio of TN to PS be $AT - AS = TS = t$, the *Module* itſelf; this Ratio is therefore diſtinguiſhed by the Title of the *Modular Ratio*, in every different Syſtem or Scale of Logarithms. What this Ratio is, in Numbers, we ſhall ſee by and by.

23. The Ordinate TN may alſo be denoted *Binomially* thus, $1+x$; its Fluxion is ſtill $\dot{x} = Mq$; and then $t \times \frac{\dot{x}}{1+x} = \dot{g}$, as before (21.) But it is now in a proper Form for computing the Fluent, or Logarithm of the whole Ratio of $1+x$ to 1, or of TN to AB.

24. For $\frac{\dot{x}}{1+x} = \dot{x} \times \frac{1}{1+x} = \dot{x} - x\dot{x} + x^2\dot{x} - x^3\dot{x} + x^4\dot{x}$, &c. (by common *Algebra*;) and by taking the Fluent of each Term, we have $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, &c. conſequently the Fluent of $t \times \frac{\dot{x}}{1+x} = \dot{g}$, is $t \times \overline{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4}$, &c. $= g = AT$, the Logarithm of the Ratio of $1+x$ to 1.

25. If the Term of the Series be taken in the Scale below Unity AB, then it will be repreſented by $1-x$, as the Ordinate ik, and its Logarithm Ai will be negative; and will be found, as above, to be $g = t \times \overline{x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4}$, &c. And the Sum of both theſe Series or Logarithms will be $2t \times \overline{x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7}$, &c. $= Ti$ the Logarithm of the Ratio of $1+x$ to $1-x$, or of TN to ik.

26. If the propoſed Ratio were that of $a+x$ to $a-x$ or $\frac{a+x}{a-x}$; then the Logarithm, or Meaſure thereof found, for each Ratio ſingly, and both added together in the Manner above exemplified, will be $2t \times \overline{\frac{x}{a} + \frac{1}{3}\frac{x^3}{a^3} + \frac{1}{5}\frac{x^5}{a^5} + \frac{1}{7}\frac{x^7}{a^7}}$, &c.

27. Hence if a = Sum of any two Quantities m and n, and x = their Difference, then will this laſt Series be the Logarithm of the Ratio of the greater (m) to the leſſer n. For ſince $a=m+n$, and $x=m-n$, we have $a+x=2m$, and $a-x=2n$; and ſo $a+x:a-x::2m:2n::m:n$. If therefore $2t\frac{x}{a}=A$, $A\frac{x^2}{a^2}=B$, $B\frac{x^2}{a^2}=C$, $C\frac{x^2}{a^2}=D$, *&c.* then $A+\frac{1}{3}B+\frac{1}{5}C+\frac{1}{7}D$, *&c.* will be the Logarithm of the Ratio $\frac{m}{n}$, or of m to n.

28. Since $t\times\overline{x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5}$, *&c.* $=g$, the Logarithm of the Ratio of $1+x$ to 1; therefore we have $\frac{g}{t}=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5$, *&c.* and therefore $\frac{g^2}{t^2}=x^2-x^3+\frac{11}{12}x^4-\frac{5}{6}x^5$, *&c.* in like Manner $\frac{g^3}{t^3}=x^3-\frac{3}{2}x^4-\frac{7}{4}x^5$, *&c.* Alſo $\frac{g^4}{t^4}=x^4-2x^5$, *&c.* and laſtly, $\frac{g^5}{t^5}=x^5$, *&c.*

29. Hence, by having the Logarithm (g) given, you find on the other Hand what Ratio it meaſures; for by *adding equal Things to Equals,* we have $\frac{g}{t}+\frac{g^2}{2t^2}=x * -\frac{1}{6}x^3+\frac{5}{24}x^4-\frac{11}{60}x^5$, *&c.* And again, $\frac{g}{t}+\frac{g^2}{2t^2}+\frac{g^3}{6t^3}=x * * -\frac{1}{24}x^4+\frac{3}{40}x^5$, *&c.* And then again, $\frac{g}{t}+\frac{g^2}{2t^2}+\frac{g^3}{6t^3}+\frac{g^4}{24t^4}=x * * * -\frac{1}{120}x^5$, *&c.* Laſtly, $\frac{g}{t}+\frac{g^2}{2t^2}+\frac{g^3}{6t^3}+\frac{g^4}{24t^4}+\frac{g^5}{120t^5}=x * * * *$, *&c.* Conſequently the Ratio ſought between $1+x$ and 1, is that of $1+\frac{g}{t}+\frac{g^2}{2t^2}+\frac{g^3}{6t^3}+\frac{g^4}{24t^4}+\frac{g^5}{120t^5}+$, *&c.* to 1.

30. If now we put $g=t$, or $\frac{g}{t}=1$, then we ſhall have the *Ratio Modularis* expreſſed thus, $1+\frac{1}{1}+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+$, &c to 1; that is, 2,71828183, &c. to 1. If this *Modular Ratio* had been derived from the Ratio of 1 to $1-x$, it would have come out 1 to $1-\frac{1}{1}+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}$, &c. or 1 to 0,36787944+.

31. If in the Series expreſſing the Logarithm 1 to $1-x$ (25), we put $x=1$, then it will become $t\times\overline{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}$, &c. $=g$; which Series being really infinite, ſhews the Logarithm Ai will be infinite when the Ordinate $ik=1-x=o$, or when the Curve meets or touches the *Aſymptote* ZY, at an infinite Diſtance.

32. Draw the Ordinate QO indefinitely near to TN, and through N and O draw Rq, Or, parallel to the Axis ZY. Then will the Area between the Curve XBW and it's Aſymptote ZY, infinitely extended towards YW, and terminated on the other Part by the Ordinate TN, be equal to the parallelogram TSRN contained under the Ordinate TN and Subtangent TS=RN. For becauſe Ns : Os :: NT : TS, it is TS×Ns=TN×Os; but TS×Ns is the Fluxion of the Parallelogram, and TN×Os is the Fluxion of the Area; and conſequently as the Fluxions are equal, the Fluents are ſo too.

33. Hence, becauſe the Subtangent is a given Quantity, that indefinite Area will always be as the Ordinate by which it is terminated.

34. In the foregoing Caſe of the *Ratio Modularis* (30) the Number 2,71828, &c. is that famous one (ſo often mentioned in the Solution of *Phyſico-Mathematical* Problems) whoſe Natural Logarithm is Unity or 1; but in the common *Briggian* Scale it is 0,43429, &c. and in the *Nautical* Scale it is 0,34377, &c. Theſe Numbers being reſpectively the MODULES of thoſe Syſtems, as we ſhall proceed to ſhew.

CHAP. III.

Of the SYSTEM *of* Natural LOGARITHMS, *derived from the* Logiſtic CURVE; *and of the* SYSTEM *of* LOGARITHMS *in the* Common CANON, *invented by Mr.* BRIGGS.

35. IN the Conſtruction of a Canon of Logarithms, it is neceſſary to inveſtigate the Logarithms of *Prime Numbers* only, the others being obtained from them by *Addition* alone. Now the Prime Numbers between 1 and 10 are four, viz. 2, 3, 5, and 7. Of theſe, and their Powers, ſuch Ratios may be compoſed, whoſe Terms differ from each but by a ſingle Unit. Thus, $\frac{2\times3^2\times7}{5^3}=\frac{126}{125}$; $\frac{3^2\times5^2}{2^5\times7}=\frac{225}{224}$; $\frac{7^4}{2^5\times3\times5^2}=\frac{2401}{2400}$; $\frac{5^4\times7}{2\times3^7}=\frac{4375}{4374}$.

36. Call the Logarithms of the Primes *a*, *b*, *c*, *d*; and let the Logarithms of the above four Ratios be called *p*, *q*, *r*, *s*, reſpectively; then by the Nature of Logarithms (7. 9.) we ſhall have $p=a+2b-c+d$; $q=-5a+2b+2c-d$; $r=-5a-b-2c+4d$; and $s=-a-7b+4c+d$. Now the Terms of the leaſt of theſe Ratios are ſo large, and their Difference ſo ſmall, that the very firſt Term of the Series (in 27) viz. $2t\frac{x}{a}$ will give the Logarithm thereof true to *Seven Places* of Figures; and the 2d Term added, viz. $2t\times\overline{\frac{x}{a}+\frac{1}{3}\frac{x^3}{a^3}}$ will give the Logarithm true to 12 Places.

37. For with reſpect to the firſt Ratio (by making $t=1$) we have $\frac{2x}{a}=0{,}0079682$, *&c.* $=p$; and in the ſame

Manner, we have $q=0,0044544$, &c. $r=0,00041658$, &c. and $s=0,0002285$, &c. Then, by the four Equations in the laſt Article you find (by the common Rules of *Algebra*) the four Logarithms a, b, c, d, of the four Prime Numbers 2, 3, 5, 7, viz. the Logarithm $a=0,6931472$; $b=1,0986123$; $c=1,6094379$; and $d=1,9459101$, &c.

38. Having obtained the Logarithms of theſe Primes; you find by Addition, the Logarithms of many other Numbers; for Inſtance,

To the Logarithm of 5=1,6094379
Add the Logarithm of 2=0,6931472

The Sum is the Logarithm of 10=2,3025851

39. By finding the Logarithm of the Primes, 11, 13, 17, 19, a much greater Number of Logarithms will yet be obtained; and this will not prove ſo difficult a Taſk as before; for put $a=10$, $x=11$, and $b=12$; then $x-1=a$, and $x+1=b$, and $x^2-1=ab$. Hence the Square of *any Prime exceeds the Product* of the two *adjacent Numbers by Unity*. Therefore, if to the Logarithm of the Ratio of the Square to the ſaid Product, you add the Logarithm of the Product, then half that Sum will be the Logarithm of the Prime*. That is, Logarithm of $\frac{x^2}{ab}$ + Logarithm of $ab=$ Logarithm x^2. the Half of which is the Logarithm of x. For Example, let $x=11$, then $x^2=121$, and $ab=120$. And the Logarithm of $\frac{121}{120}=0,0082988$ (27), to which add the Logarithm of $120=4,7874917$, the Sum is 4,7957905, which is the Logarithm of x^2; and the Half of that, viz. 2,3978952 is the Logarithm of $x=11$, as required. After the ſame Manner the Logarithms of other Primes are found, and from them the Logarithms of all other Numbers.

40. The Logarithms hitherto treated of are properly called *Natural Logarithms* whoſe Module (t) is Unity.

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* Becauſe $\frac{x^2}{ab}\times ab=x^2$.

But this Form or Syſtem of Logarithms (though very ſerviceable in many Parts of Mathematics) was ſoon diſcovered to be not ſo commodious and eaſy in *Trigonometrical Computations* as another Form or Canon of Logarithms would be, wherein the Logarithm of 10 conſiſted only of *Unity* with Cyphers annexed, as 1,0000000, *&c.* Becauſe, wherever the *Radius* is concerned, as it is always ſuppoſed to conſiſt of 10000000000 equal Parts, ſo its Logarithm will be 1,0000000, &c. And conſequently by this contrivance, only one *Addition* or one *Subtraction* will be required, inſtead of both Operations together.

41. This Alteration in the Form of Logarithms we owe to our countryman Mr. *Henry Briggs*, and it is therefore called the *Briggian Canon*, and is that now in common Uſe. It has been ſhewn, that in every Syſtem of Logarithms, the Modules are as the Logarithms of any given Ratio (21), and therefore it will be as the Logarithm of 10, viz. 2,3025851 is to the Module 1 (in the *Natural Syſtem)* ſo is 1,0000000 (the Logarithm of 10) to 0,434294482 = Module of *Briggs's Syſtem*; and this latter Module, it is evident, is only the *Reciprocal* of the former.

42. On the Contrary, by having given the Modules of any two Syſtems or Scales of Logarithms, and the Logarithm of a Ratio in one, you find the Logarithm of the ſame Ratio in the other; for the Module of one is to the Module of the other, as the Logarithm in one is to the Logarithm of the ſame Ratio in the other.

43. We have now ſhewn the Conſtruction of two Syſtems of Logarithms, viz. of the *Natural Logarithms* invented by LORD NEPER, a *Scotch* Baron, and publiſhed in the Year 1614; and the *Briggian Syſtem* in common Uſe. But there is a third Syſtem of Logarithms invented and publiſhed by our Countryman Mr. ED. WRIGHT, before the Year 1610; and is therefore the *firſt* SYSTEM of LOGARITHMS that was ever invented; but as theſe

Logarithms are of the *Nautical Kind*, and deſigned immediately for the *Mariner's Uſe*, we ſhall next explain the Principles from which they ariſe; and that in a Manner not to be met with in any Books on NAVIGATION hitherto publiſhed.

CHAP. IV.

Of the Firſt, *or* Original SYSTEM *of* LOGARITHMS *invented by* Mr. EDW. WRIGHT; *and their* USE *in conſtructing the* Nautical PLANISPHERE *for* MERCATOR's SAILING.

44. IN the *Nautical* PLANISPHERE, uſually called *Mercator's Chart*, the Meridians being all right Lines parallel to each other, and the Radius of every Parallel of Latitude equal to that of the Equator, it follows, that the Degrees in every ſuch Parallel will be enlarged in this Chart in the Ratio of the *Radius of the Equator to the Radius of the Parallel*, that is, *in the Ratio of the Radius to the Co-ſine of Latitude.*

45. In order then to preſerve the ſame reſpective Poſition and Bearing of Places in this Chart, as they have upon the Globe, it is neceſſary that the Degrees of Latitude ſhould be every where enlarged in the ſame Proportion, viz. *of Radius to the Co-ſine of the Latitude*; and then the Meridian conſiſting of ſuch enlarged Degrees, is called the *Nautical* MERIDIAN, or (upon the common Scales) the *Line of Meridional Parts.*

46. Theſe *Meridional Parts* are Numbers contained in the Tables of that Title in modern Books of Navigation, copied from the original Table compoſed by Mr. *Edward Wright* under the Title of a *Table of Latitudes*, &c. That is, a Table of Numbers expreſſing the Length of an Arch of the Meridian, conſiſting of theſe enlarged De-

grees to any Latitude; and theſe Numbers make the *Nautical* SYSTEM of LOGARITHMS, the Principles of which we are next to explain.

47. Let AED be a Meridian of the Globe (Fig. 2.); C the Center; EC the Radius of the Equator; BG the Radius of a Parallel in the Latitude EB; and draw BC, BD, BA; and AS indefinitely near to AB; alſo through any other Point M in the Meridian, draw AL to interſect the Radius CE (continued out if need be) in L; and to the Point B draw the Tangent BH. Laſtly, at the Equator E erect the Perpendicular ER, in which let EQ be the Part of the *Nautical Meridian* correſponding to the Natural Arch of Latitude EB.

48. Now put the Natural Arch $MB=z$; the correſponding Nautical Line $OQ=x$; and $CN=y$, the Tangent of half the Co-Latitude BD. Then will the very ſmall Particle $SB=\dot{z}$, $PQ=\dot{x}$; and $FN=\dot{y}$, the Fluxion of the Ratio of CN to CL. Alſo let SX be drawn parallel to BG, cutting AB in Z. Then is $SB=SZ$, for the Angles $ABG+GBD=ABH+ABC=90°$. but becauſe $AC=CB$, therefore $ABC\ (=BAC)\ =GBD$; therefore $ABG=ABH=AZX=SZB$, that is $SBZ=SZB$, and conſequently $SB=SZ$.

49. Then to find the *Nautical Magnitude* OQ of any Arch MB, we proceed thus; it has been ſhewn that $OQ : SB :: \dot{x} : \dot{z} ::$ Radius CE, Co-Sine of Latitude BG or SX, that is, $OQ : SZ :: CE : SX$. But $SZ : SX :: FN : NC$; therefore $OQ : CE :: FN : CN$. That is, in Species, $\dot{x}$: Radius :: $\dot{y} : y$; and conſequently, Radius $\times\frac{\dot{y}}{y}=\dot{x}$. Therefore $\dot{x}$ is the Meaſure or Logarithm of the Ratio $\frac{\dot{y}}{y}$ to *Radius*, as a *Module*. (See Art. 20.)

50. Hence it follows, ſince Fluents are in the ſame Ratio with their contemporary Fluxions; that the Nautical Magnitude OQ of the Arch MB (the Fluent of $\dot{x}$) is

the Logarithm of the Ratio of CN to CL (the Fluent of $\frac{\dot{y}}{y}$) to the *Module* or *Radius* CE of the Sphere. And thus we have proved that the *Nautical Meridian Line*, is the ſame with the Line of *Artificial Tangents*; or that the Table of *Meridional Parts*, is really *a Table of Logarithms of the Ratios of the Tangents of the Half Complements of the Latitudes (of any two Parallels) to the Radius of the Globe, as the* MODULE *of this Nautical Syſtem.*

51. If the Point M coincides with E, then we have the Nautical Magnitude of the Arch EB equal to EQ, which is the Logarithm of the Ratio of CN to CE or Radius; and therefore, for any Arch, beginning at the Equator, the *Nautical Magnitude* or *Meridional Parts* will be the Logarithm that meaſures the Ratio of the Tangent of half the Co-Latitude of that Arch to the Tangent of 45° or Radius.

52. For Example, ſuppoſe the Latitude EB=50° the Complement is BD=40°, the Half of which is 20° =DI; and the Tangent thereof is DK=NC, the Ratio of which to CE Radius is, in the common Canon of *Briggs*'s Logarithms, thus found.

From the Logarithm Tangent of 45°	10,0000000
Subduct the Logarithm Tangent of 20°	9,5610659
There remains the Logarithm of the Ratio of CN to CE, - - - - - - viz.	0,4389341

Which Logarithm 0,4389341 will therefore expreſs the *Nautical Magnitude*, or be the Meridional Parts for the Arch of 50°=EB, in the *Briggian Syſtem.*

53. In every Syſtem of Logarithms the Modules are to each other as the Logarithms of the ſame Ratio (21.); ſince then in *Briggs*'s Logarithms the Module is 0,434294482, *&c.* and the Logarithm of the Ratio of DK or CN=3639702 to Radius CE=1000000 is 0,4389341; and the Logarithm of the ſame Ratio in

Wright's Table is 0,3474604; therefore ſay, As 0,4389341 : 0,3474604 : : 0,434294 : 0,34377468 which is therefore the Module of *Wright's Syſtem of Logarithms* or *Table of Nautical Latitudes.*

54. In every Circle there are 360°. and of courſe, 21600 Minutes; in the Radius of the Circle there are therefore 3437 ¾ Minutes, which is the very ſame Number as the Module of *Wright's Syſtem*; and therefore this Module conſiſts of Minutes of the Equator, which are therefore to be conſidered as *Logarithms* (as well as *Minutes*) of *Longitude*, upon a Courſe or *Rhumb* of 45 Degrees.

55. For the MODULE is itſelf the Logarithm of the *Modular Ratio*, viz. of 1 to 0,367879, *&c.* (30.) that is, in the preſent Caſe, of the Ratio of the Tangent of 45° = 1, to the Tangent = 0,367879, *&c.* which in the Table of Natural Tangents ſtands againſt 20°. 11′. 43″; the Double of which is 40°. 23′. 26″. and this is the Co-Latitude of 49°. 36′. 34″. which is therefore the Latitude made on Sailing upon the Rhumb of 45°. to gain a Longitude = 3437 ¾ Minutes = 57°. 17′. 44″. Now in Mr. Wright's Table of Nautical Latitudes, againſt the Latitude 49°. 36′. 34″. you find the very ſame Number of Meridional Parts, *viz.* 34377468 for the Length of the *protracted Meridian* correſponding to that Latitude. Therefore *the Longitude in the Equator is the Nautical Magnitude of the Latitude (or Difference of Latitude) made upon a Courſe of* 45°, or equal *to the Line of Meridional Parts agreeing to that Latitude.*

56. Of this we have alſo a direct Demonſtration from the Principles of the *Stereographic Projection*; for let EPC (Fig. 3.) be the 8th Part of the Globe ſo projected *in Plano*; P the Pole; EC the Equator; EP, PC, Meridians; and E*e*L the Rhumb of 45° here projected into the *Equiangular Spiral*, cutting all the Meridians in the ſaid Angle of 45°. Now let BG be a Parallel of Lati-

tude cutting the Spiral in (*c*), through the ſame Point *c* draw the Meridian PA; laſtly, draw the Meridian P*a* indefinitely near to PA, and the Parallel SX to BG; and there will be formed the indefinitely ſmall Triangle *cde*, in which the Sides *ed* and *cd* are equal; but *ed* is to A*a* *as Co-ſine of the Latitude* A*c* *to the Radius of the Globe*; therefore A*a* is the Nautical Magnitude of *ed*, or of *dc*=BS; therefore the Sum of all the Particles A*a*, or the Arch EA, is the Nautical Magnitude of the Sum of all the Particles *dc*, or the Latitude A*c*=EB, which was to be ſhewn. (See 44, 45,).

57. Hence the Reaſon why, in the Solution of all Caſes of *Mercator*'s SAILING, the Meaſures of the Sides of the *Nautical Triangle* are always expreſſed in Minutes or Miles of the Equator; the Table of *Meridional Parts*, or *Nautical Logarithms*, being the very ſame Thing.

58. Having found the MODULE of WRIGHT'S SYSTEM of LOGARITHMS, we find the Logarithm of 10 (or of 10 to 1) thus; as *Brigg*'s Module 0,434294 to 1, (his Logarithm of 10;) ſo is *Wright*'s Module 0,34377468 to 0,7915704, &c. which is therefore the Logarithm of 10, in his Scale. As we have now inveſtigated the Nature and Genius of the Three principal Syſtems of Logarithms in Uſe, we ſhall here place their reſpective Characteriſtics in one View, and in the Order of Time they were invented and publiſhed.

	Modules.—	—Log. of 10.
Wright's Nautical Logarithms	0,34377468—	0,79157045
Neper's Natural Logarithms	1,00000000—	2,30258509
Briggs's Common Logarithms	0,43429448—	1,00000000

59. We conclude this Theory of *Wright's Logarithms* by obſerving one Particular more, viz. that his *Nautical Meridian* is the very ſame Thing as the *Common Line of Artificial Tangents*, begining from 45°, or Radius; and

numbered 10, 20, 30, 40, &c. inſtead of 5, 10, 15, 20, &c. the Reaſon of which will be evident when we conſider that the Nautical Latitude of 10° is the Logarithm of the Ratio of the Tangent of 45° to that of 40. And the Nautical Latitude of 20° is the Logarithm of the Ratio of the Tangent of 45° to that of 35°; and ſo on.

60. And here we ſhall find *Briggs*'s Logarithm of the Tangent of 45° to that of 40°, is in the ſame Ratio to *Wright*'s, as the Module of the Common Canon to the Module of the Nautical Canon, or Meridional Parts.

For the com. Log. of the Tangent of 45° is 10,0000000
From which take the Log. Tangent of 40 9,9238135

There remains the Ratio - - - 0,0761865

But we have 0,4342945 : 0,3437747 : : 0,0761865 : 0,0603047; which is the Number anſwering to the *Nautical Latitude of* 10 *Degrees* in Mr. *Wright*'s original Table. And therefore, fully proves the *Nautical Meridian Line, and Line of Artificial Tangents from* 45°, *are the ſame Thing.*

C H A P. V.

Of the Nautical MERIDIAN LINE, *and* Logarithmic SCALES, *and* MODULES *of the Three foregoing* SYSTEMS, *explained in a large* Nautical DIAGRAM.

61. IN the large Diagram of Fig. 4. is contained a Synoptic View of the whole Doctrine of Logarithms delineated; in which C is the center of the EARTH; AGD a Quarter of a Great Circle divided into 90 Degrees; BD is a Line of *Natural Tangents*, divided as far as 45°; AC is a Line of *Natural Numbers*, divided into 10,000000 equal Parts in the common Canon.

62. In the Radius or Line CD, continued out, take CE to CD as the Logarithm of 10 to the Module of the Syſtem (of any Sort;) and make the Perpendicular EF =1; then will the Line CE be the Logarithm of the Ratio of AC to EF, or of 10 to 1; and is always divided into an indefinite Number of equal Parts, according to the Module of the Syſtem. In our common Tables, this Number is 1,0000000, &c.

63. This Line CD continued out towards I, is a Line of *Nautical Latitudes*, or *Meridional Parts*, which are Logarithms of the ſeveral Ratios of Radius AC to the Tangents of the half Co-latitudes; thus C 10 is the Logarithm of the Ratio of AC, or DB to D 40; and C 20 is the Logarithm of the Ratio of DB to D 35; and ſo on. Therefore, if in the Points 10, 20, 30, 40, &c. there be erected Perpendiculars *a* 10, *b* 20, *c* 30, &c. equal to the Tangents D 40, D 35, D 30, &c. to D 5=80 *h*, then will the Points A, *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, &c. be all in the *Logarithmetic Curve* A *e* H.

64. Draw the Chord of 90°. AD; that will be the Tangent of the Curve at A; and the Radius CD will be the *Subtangent* or Module of the Syſtem; and D*e* to DB is the *Ratio Modularis*. Alſo the *Modular Rectangle* (32) is here the Square ABCD, equal to the infinite Area of the curvelineal Space HACI.

65. Next to the *Line of Meridional Parts* CI, is placed the *Line of Longitudes* KL, which, upon Navigation Rules, is called the *Line of equal Parts*. By this, the Degrees and Minutes of Longitude anſwering any Latitude (or Difference of Latitude,) made by ſailing on a Rhumb AD=45° is eaſily ſeen, and *vice verſa*.

66. The Line MN is the common Scale of *Artificial Tangents*, which was ſhewn to be the ſame, and ſimilarly divided with the *Line of Latitudes* CI, but only numbered

differently; *one Degree* here being every where equal to *two* there, which is thus demonſtrated; upon the Center A (Fig. 2.) deſcribe the Arch Cc, which cuts the Lines AN, AF in the Points a, b; alſo with the Radius AN deſcribe the ſmall Arch Ne; and let the Radius $AC=r$, the Arch $Cc=z$, and $ab=\dot{z}$; $CN=y$, and $NF=\dot{y}$ as before. Then becauſe of ſimilar Triangles ACN, and NeF, we have $FN : Ne :: AN : AC$; but it is $Ne : ab :: AN : Aa=AC$; therefore, $FN : ab :: AN^2 : AC^2$; that is, $\dot{y} : \dot{z} :: rr+yy : rr$; conſequently, when CN the Tangent becomes CE the Radius, then $y=r$, and $\dot{y} : \dot{z} :: 2 : 1$; or, in that Caſe, $\dot{y}=2\dot{z}$. Again, the Fluxion of the Logarithm of any Tangent y, is $\frac{\dot{y}}{y}$ (by Article 20.) therefore when $y=r=1$, the ſaid Fluxion of the Logarithmic Tangent is $\dot{y}$, the ſame as the Fluxion of the Natural Tangent, and equal to $2\dot{z}$; but $\dot{z}=\dot{x}$, the Fluxion of the Nautical Meridian Line at the Equator C; therefore, $\dot{y}=2\dot{x}$; therefore, a Degree of *Briggs*'s Logarithmic Tangents is every where equal to two Degrees of *Wright's Nautical Meridian*, at the ſame Latitude.

67. The Line OP is the *Nautical Scale or Syſtem of Logarithms*, by Mr. WRIGHT, conſiſting of *Minutes of the Equator*. The *Module* of this Syſtem is the Subtangent or Radius CD=3437,747 in this Line, as is evident by Inſpection (53, 54.) And the Ratio of the Tangent of 45°=10=AC to the Tangent of 1=EF, is 7915,704; in the common Canon that Ratio is 10,000. Moreover, any Arch or Number of Degrees of Longitude in the Line KL, being applied to this *Line*, the *Number of Minutes or Miles* of the Equator contained therein, will inſtantly appear.

68. The Line QR is the Scale of *Common Logarithms* by Mr. *Briggs*; in which the *Module* is the Subtangent CD, anſwering to 4342,945 in this Line (41.) Alſo

the Logarithm of the Ratio of AC=10 to EF=1, is 10000,000, &c. (40.) As the Radius AC may be conſidered as divided into 10,0000000, &c. equal Parts, ſo the Logarithm of any Natural Number is found in this Scale by meaſuring of the Ordinate equal to it from the Point E. For Example, the Number 8391 in the Line AC is equal to the Tangent of 40°; the ordinate repreſenting it is *a* 10; then the Logarithm E 10, being applied in the Line QR, will be found 9238,135, as you ſee it in the common Tables. So the Diſtance of the Ordinates equal to the Digits 2, 3, 4, &c. from E meaſured upon QR will be found 3010,300; 4771,213; 6020,600, &c. as in the Table.

69. The Line ST is the Scale of *Natural Logarithms*, immediately reſulting from the *Logiſtic Curve* itſelf, viz. A*f*H. In this Syſtem, we have ſhewn the *Module* or *Subtangent* CD=10,000; and the Logarithm of AC to EF, or of 10 to 1, is 23025,851 (ſee Art. 38.) And thus the Diſtance of any Ordinate =2, 3, 4, &c. taken from E, and meaſured on this Scale ST, will ſhew the *Natural Logarithm* of that Number to be 6931,472; 10986,123; 13862,943; and thus you proceed for any other Number.

70. The Line VW is called the *Line of Numbers*, and was invented by Mr. ED GUNTER, and is therefore called *Gunter's Line*, or, ſimply, the *Gunter*; it contains the Natural Numbers as they ſtand in the Scale of Proportionals between 10 and 1; ſome of it's remarkable Uſes we ſhall ſee hereafter; by which it will appear how eaſy it is to find the Logarithm of any propoſed Number in any one of the *Three* SYSTEMS (and *vice verſa*), and that to ſo conſiderable a Degree of Exactneſs, as will render them of no ſmall Uſe and Importance in all the Branches of *practical Mathematics*, and of *Navigation* in particular.

CHAP.

C H A P. VI.

The Rationale *of the* PROCESSES *previouſly neceſſary to the* APPLICATION *of* LOGARITHMS *in the various* COMPUTATIONS *of* ARITHMETIC *and* TRIGONOMETRY.

71. FROM the Nature of Logarithms, it is eaſy to ſee they only indicate what Power of 10 any Natural Number is; thus in the following

$$\text{Series}\left\{\begin{array}{llllll} 0. & 1. & 2. & 3. & 4. & 5.\ \&c. \\ 1. & 10. & 100. & 1000. & 10000. & 100000.\ \&c. \end{array}\right.$$

the Logarithm 2 ſhews that 100 is the 2d Power of 10; and 3 ſhews 1000 to be the 3d Power of 10; and ſo on in the reſt: Therefore, the Logarithms in the Table againſt the Number 3 and 30, ſhew that $3=10^{0,477121}$ Power of 10; and that $30=10^{1,477121}$ Power of 10; alſo that $3045=10^{3,483587}$ Power of 10.

72. From hence the Nature of the *Indices* of Logarithms appears; for the *Index* is that Part or Figure on the Left Hand ſeparated by a Dot or Comma from the reſt, and as it is the *integral* Part of the Logarithm, ſo it denotes the Number of *Integers* or Places of whole Numbers, leſs one, in the Number to which the Logarithm belongs; thus, if the Index be 3, it ſhews the Number has 4 Integers; the Index 2 ſhews the Number has 3 integral Places; and ſo on. And while there are any Integers in the Number, the Index is ſaid to be *poſitive*; but *negative*, when it belongs to a Number, wholly a *decimal Fraction.* If the Decimal has no Cypher prefixed, the negative Index is .9. if it has one Cypher, the Index is .8. if 2 Cyphers, the Index is .7., and ſo on. But this will be beſt underſtood by the following Table of Numbers, Logarithms, and their poſitive and negative Indices. Note, the negative Indices are dotted on each Side to diſtinguiſh them from the poſitive ones.

Number.	Logarithms.	Number.	Logarithms.
62060	4.792812	0.6206	.9.792812
6206	3.792812	0.06206	.8.792812
620.6	2.792812	0.006206	.7.792812
62.06	1.792812	0.0006206	.6.792812
6.206	0.792812	&c. &c.	

73. It ſometimes happens, that the Logarithm of a Number, which is not contained in the Table, may be required. As ſuppoſe it required to find the Logarithm of the Number 5423758; this Number lies between the Numbers 5423700 and 5423800. Now then put this intermediate Number $5423758 = m$, and the next leſs $5423700 = n$; then we have $m + n = a = 10847458$; and $m - n = x = 58$. In the common Tables $t = 0,4342945$; and $2t\frac{x}{a} = 0,0000046$, &c. which is the Logarithm of the Ratio of m to n (27.) or of the Number 5423758 to 5423700; therefore,

To the Logarithm of 5423700 - - - 6.7342957
Add the Logarithm juſt found - - - 0,0000046

The Sum is the Logarithm ſought of 5423758 — 6.7343003

74. There is a Method of finding this Logarithm by *proportional Parts* in *Sherwin*'s, and other large Tables, with Examples of the Operation; the Reaſon of which is eaſy to conceive from Fig. 1. for therein let TN be the intermediate Number, and OQ, ML the adjacent ones; and let $NT - OQ = Ns$, $ML - OQ = Mr$, and $ML - NT = Mq$. Alſo the Differences of the Logarithms of theſe Numbers are $AT - AQ = TQ = Os$, $AL - AQ = QL = Or$, and $AL - AT = TL = Nq$. Now the Triangles NsO and MrO being ſimilar, give $Mr : Or :: Ns : Os$; that is, *the Difference of the adjacent Numbers is to the Difference of their Logarithms, as the Difference of the intermediate and leaſt adjacent Number, is to the Difference of*

their Logarithms, which Difference added to the Logarithm AQ, gives the Logarithm AT required.

75. The Converſe of this finds the *Number to any Logarithm propoſed*; for then we have Or : Mr :: Os : Ns; which Difference Ns, added to the leſſer adjacent Number OQ, gives NT the intermediate Number required. This may alſo be found as above (73.) by putting the given Logarithm of OQ to NT, viz. Os or QT $= g = 2t \frac{x}{a}$; for then we have $g = \frac{2tx}{2m-x}$ or $\frac{2tx}{2n+x}$; and therefore, $x = \frac{g}{t+\frac{1}{2}g}\, m$, or $\frac{g}{t-\frac{1}{2}g}\, n = \mathrm{N}s$, which added to OQ gives NT, the Number ſought. In the Example above $g = 0{,}0000046$; $t = 0{,}4342945$; $m = \mathrm{LM} = 5423800$ and $n = \mathrm{OQ} = 5423700$; whence $\mathrm{N}s = x = 58$, which gives TN = 5423758, the Number required of the given Logarithm AT = 6,7343003.

76. In ſuch Computations, where you have a Logarithm to ſubduct from another, you add it's *Arithmetical Complement* to that other Logarithm, and the Reſult is the ſame. This *Arithmetical Complement* is the Difference between any Logarithm and 10,0000000, or rather 9,9999999, &c. For Example:

From the Logarithm - - -	9.9999999
Subduct the Logarithm of 525 - - -	2.7201593
Remains the *Arithmetical Complement*	7.2798406
To which add the Logarithm of 20 -	1.3010300
The Sum is Logarithm of $\frac{20}{525}$ = 0,03895 —	8.5808706
The ſame as if from Logarithm of 20 -	1.3010300
You ſubduct the Logarithm of 525 -	2.7201593
The Quotient is, 0,03895 - -	8.5808707

77. In *Sherwin*'s Tables you find the Logarithm *Sines* and *Secants* placed by each other, as alſo the Logarithm

Tangents and Co-tangents; and the Reaſon is becauſe they *are mutually Arithmetical Complements* to each other; ſo that if a Sine is to be *ſubducted* you may *add* the oppoſite Secant; or if a Tangent is to be *ſubducted* you *add* the Co-tangent (and *vice verſa*) to anſwer the ſame End; which is very commodious to thoſe who are much uſed to Logarithmic Calculations. This Artifice is founded on the Conſideration, *That Radius* : *Sine* : : *Secant* (*of the ſame Arch*) : *Radius*. And again, that *Radius* : *Tangent* : : *Co-tangent* : *Radius*; for Proof of which, ſee my *Young* TRIGONOMETER'S *New* GUIDE.

C H A P. VII.

Of the OPERATIONS *of* Common ARITHMETIC *by* LOGARITHMS.

78. THERE is no Difficulty in the OPERATIONS of ARITHMETIC by LOGARITHMS, but in what regards the *Indices* for *Decimal Numbers*, in rightly ordering them for the various *Sums*, *Differences*, *Products*, and *Quotients*, which ariſe in the ambiguous Proceſſes of this Kind. As the Indices for *Integers* all begin from (0) and ariſe in the Series 1. 2. 3. 4. 5. &c. ſo thoſe for Decimals might be expected to deſcend below (0) in the ſame Order, .1. .2. .3. .4. .5. &c. only dotted on both Sides to diſtinguiſh them from the other. But in this Caſe, theſe Indices muſt be conſidered as really *negative*, and being to be *added*, *ſubtracted*, &c. with thoſe which are *poſitive*, will require all the Rules of *Algebra* to be conſtantly obſerved, with regard to the Change and Alteration of the Signs + or —, as there uſed.

79. To avoid this Trouble, the Logarithm of Unit or 1, inſtead of (0) is ſuppoſed to be 10; and then the Series below for Decimals will be .9. .8. .7. .6. &c. as at (72.)

and theſe Indices only ſhew how many Cyphers are to be prefixed to the ſignificant Figures of the Decimal Number, as there declared. But when the Number of theſe Cyphers exceeds 10, it is troubleſome, in this Notation, to determine them, eſpecially for Learners. And therefore the beſt way, in ſuch Caſes, is to make the Logarithm of 1 = 100, and the Indices for the Decimals will then be this decreaſing Series, viz. .99. .98. .97. .96. &c. ſhewing the Number of Cyphers to be prefixed are ſo many as the given Index is leſs than 99. And, for the Sake of Beginners, we give the following Examples of the ſeveral Rules.

80. Examples of MULTIPLICATION by LOGARITHMS.

Whole Numbers.

Multiply	127—2,103804	9876—3,994581
By - - -	12—1,079181	517—2,713490
The Product	1524—3,182985	5105892—6,708071

81. *Mixed Numbers.*

Multiply	7,5—0,875061	0,762—.9.881955
By - -	10—1,000000	570—2.755875
Product	75—1,875061	434,34—2.637830
Multiply	36,5—1,562293	0,00001—.5.000000
By -	0,00019—.6.278754	100000—5,000000
	0,006935—.7.841047	1—0,000000

82. *Pure Decimals.*

Multiply	0,12—.9.079181	0,0097 — .7.986772
By - -	0,8 — .9.903090	0,00021 — .6.322219
Product	0,096 — .8.982271	0,000002037-.4.308991

Multiply

Multiply - -	0,00000000075	—.90.875061
By - - - -	0,000006	—.94.778151
	0,0000000000000045	—.85.653212

83. Examples of DIVISION by LOGARITHMS.

Whole Numbers.

Divide	1524—3,182985	5105892—6,708071
By	172—2,103804	517—2,713490
Quotient	12—1,079181	9876—3,994581

84. *Mixed Numbers.*

Divide	75—1,875061	434,34—2,637830
By -	7,5—0,875061	570—2,755875
Quotient	10—1,000000	0,762—.9.881955

Divide	0,006935—.7.841046	1 — 0,000000
By - -	36,5— 1.562293	0,00001 —.5.000000
	0,00019—.6.278753	100000 — 5.000000

85. *Pure Decimals.*

Examples. Divide - - -	0,000002037	—.4.308991
By - - - -	0,0097	—.7.986772
Quotient - -	0,00021	—.6.322219

Divide	0,0000000000000045	—.85.653212
By - - -	0,000006	—.94.778151
Quotient -	0,00000000075	—.90.875061

86. As the RULE of THREE conſiſts only of one *Multiplication*, and one *Diviſion* (12), there is no farther Variety in the Operation by Logarithms; we paſs therefore to the *Involution* and *Evolution* of Powers and Roots of Numbers, in which Operations, Logarithms are of moſt ſingular Service.

87. IN-

87. INVOLUTION by LOGARITHMS (9.)

Examples. Required the *Square* of 32 = 1,505150
Multiply by the Index of the Power - - 2

Product is Log. of the Square required 1024—3,010300

Required the *Cube* of - - - - - - 32—1.505150
Multiply by the Index - - - - - - 3

Logarithm of the Cube - - - 32768—4.515450

Required the *Surſolid* or 5th Power of 32—1.505150
Multiply by the Index - - - - - - 5

Logarithm of the 5th Power - 34554432—7.525750

Required the 9th Power of - - 1,05—0,021189
Multiply by the Index - - 9

The 9th Power - - - 1,516—0,180701

Required the Square of - - 0,32—.9.505150
Multiply by - - - 2

The Square of - - 0,32=0,1024—.9.010300

Required the $\frac{3}{2}$ Power of - - 32— 1,505150
Multiply by the Index $\frac{3}{2}$=1,5 - - 1,5

The Square required - - 181,03—2,257725

Required the Cube of - - 0,0012—.98.079181
Multiply by - - - - - 3

The Cube required - 0,000001728—.94.237543

88. EVOLUTION *by* LOGARITHMS.

Examples. Required the *Square Root* of 1024—3,010300
Divide by the Index of the Root 2

Quotient is the Square Root - 32—1,505150

Required the *Cube Root* of - 32768—4,515450
Divide by - - - - - 3

The Cube Root - - 32—1,505150

Required the 9th Root of - 1,516—0,180701
Divide by - - - - - 9.

The Root required - 1,05—0,021189

Required the $\frac{3}{2}$ Root of - 181,03—2,257725
Divide by $\frac{3}{2}$ - - - - - =1,5

The Root required is - - 32—1,505150

Required the Cube Root of 0,000001728—.94.237543
Divide by - - - 3

The Root ſought - - 0,0012—.98.079181

89. There is but one Thing that is puzzling to the Learners of this Science, in the above Operations, and that is, why the 10's or 100's are to be rejected from the Indexes of Products by Multiplication and Involution; and the ſame to be added to thoſe we are to divide. But the Reaſon of all this will appear, when the Matter is properly conſidered in Fig. 1. where if AB be Unity, and its Logarithm 10, or 100; then the Logarithms of the Decimal Proportionals below AB, as of *cd*, *ef*, *gh*, *ik*, &c. will be .9. .8. .7. .6. &c. or .99. .98. .97. .96. &c.

90. Therefore it will be AB : *cd* : : *ef* : *gh*; and ſo AB × *gh* = *cd* × *ef*; and ſo in Logarithms 10 + .7. = .9. + .8. therefore, .9. + .8. − 10 = .7. Or, 100 + .97. = .99. + .98. and .99. + .98. − 100 = .97. From whence the Reaſon of rejecting 10, or 100, from the Indices of Products in *Multiplication* is evident.

91. In Diviſion, becauſe $\frac{AB \times gh}{ef} = cd$, therefore the Logarithms $10 + .7. - .8. = .9.$ and conſequently the Reaſon why 10 is added to the leſſer Index .7. is very plain.

92. Again, becauſe $AB : cd :: cd : ef$, and ſo $AB \times ef = cd \times cd = cd^2$; therefore $10 + .8. = .9. + .9. = 2 \times .9.$ conſequently $2 \times .9. - 10 = .8. =$ Logarithm of ef, which is the Square of cd. Moreover, ſince $ed : ef :: ef : gh$, it is $ed \times gh = ef^2$, then in Logarithms $.9. + .7. = 2 \times .8.$ therefore $\frac{.9. + .7.}{2} = .8. = 2 \times .9. - 10$; conſequently $.9. + .7. = 4 \times .9. - 20$, therefore $3 \times .9. - 20 = .7. =$ Logarithmic Index of gh, which is the Cube of cd. Hence in *Involution*, you reject as many 10's as there are Units in the Index of the Power leſs 1.

93. In *Evolution*, juſt the contrary is to be done; for ſince we have $AB \times ef = cd \times cd$, it is $\sqrt[2]{AB \times ef} = cd$; therefore, the Logarithms $\frac{10 + .8.}{2} = .9. =$ the Logarithm of the Square Root of the Decimal ef. Again, $AB \times gh = cd \times ef$; and ſo $10 + .7. = .9. + .8.$ But $10 + .7. - .9. = .8. = 2 \times .9. - 10$; therefore, $2 \times 10 + .7. = 3 \times .9.$ And conſequently, $\frac{20 + .7.}{3} = .9. =$ Index of the Logarithm of cd which is the Cube Root of the Fraction gh. Hence the Addition of as many 10's to the Index of the Logarithm of the given Power, as there are Units in the Index of the Root leſs 1.

CHAP. VIII.

The SOLUTION *of the* CASES *of* Mercator's SAILING *by the* Table *of* Logarithmic TANGENTS *in the* CANON *in* Common USE.

94. FROM the foregoing Theory of Logarithms it will be eaſy to ſee, that a Solution to all the Caſes of *Mercator*'s SAILING may be readily had from the Common Tables of *Artificial Tangents*; for which Reaſon, the Table of *Meridional Parts* was thought unneceſſary, and therefore left out of *Sherwin*'s COLLECTION of *Mathematical* TABLES.

95. For, to ſhew this, nothing further is requiſite than only to demonſtrate, *that the Differences of Longitude made in ſailing upon two different Courſes, will be proportional to the Tangents of thoſe Courſes.* To this End, let PE (Fig. 5.) be a Meridian projected into Radius; EHQ Part of the Equator; PH a Meridian, in which a Ship ſails from the Latitude L, to the Latitude M in the Meridian PI, upon the Rhumb ELW of 45°; in this Courſe, the Difference of Latitude is LO, and that of Longitude HI, which is alſo the Logarithm of the Ratio of LP to MP, or of the Tangents of half the Co-Latitudes of the Ports L and M. (50.)

96. Suppoſe another Ship ſails from the ſame Port L, but upon another Rhumb L *m w* to a Port *(m)* in the ſame Latitude or Parallel O M *m*; ſo that ſhe makes the ſame Difference of Latitude LO, as before; but the Difference of Longitude is now H*i*.

97. If now L O be conſidered as *Radius*, then MO will be the Tangent of the Angle MLO; and *m*O the Tangent of the Angle *m* L O. But the *Meridional Dif-*

ference MO : *m*O : : HI : H*i* *Therefore the Difference of Longitude* HI *is to* H*i*, *as the Tangent of the Courſe on the Rhumb* LW, *to the Tangent of the Courſe on the Rhumb* L*w*.

98. In *Wright*'s Syſtem of Logarithms, the *Module* of the Syſtem CD (Fig. 4.) is the *Tangent* of the Angle of the Courſe CAD (=45°) upon which the Differences of Longitude are the Logarithms of the Ratios of the Tangents of the Half Co-Latitudes (50.) For the ſame Reaſon, in any other Syſtem of Logarithms, the Module of that Syſtem will be the Tangent of a certain Angle or Courſe, upon which the Differences of Longitude will alſo be the Logarithms of that Syſtem; and every where expreſs the Ratio of the Tangents of the Co-Latitudes of the Ports the Ship departs from, and arrives at.

99. Therefore, to find this Angle in *Brigg*'s Syſtem, or the common Tables of artificial Tangents, we have this Analogy, viz. As the Module in *Wright*'s Syſtem 0,3437745 is to the Tangent of the Courſe 45°, ſo is the Module of *Brigg*'s Syſtem 0,4342945 to the Tangent of the Angle 51°. 38′. 9″. And conſequently, if a Ship ſails upon the Rhumb, which makes an angle of 51°. 38′. 9″. with the Meridians it paſſes, then the Differences of the Logarithmic Tangents (in the common Tables) of the half Complements of the Latitudes, will be the Differences of Longitude made on ſuch a Courſe.

100. Hence, we derive Solutions to all the Caſes of *Mercator*'s SAILING, wherein the Longitude is concerned from the common Tables of artificial Tangents. For (by 97 and 99) we have this Analogy, *As the ſaid Difference of Longitude (or of the common Logarithmic Tangents of the half Co-Latitudes of two Ports) made on the Rhumb of* 51°. 38′. 9″. *is to the Difference of Longitude made in ſailing on any other Rhumb between the ſame Parallels, ſo is the Tangent of* 51°. 38′. 9″. *to the Tangent of the Courſe on that Rhumb.*

101. Put
A = Difference of the Logarithmic Tangents of the half Co-Latitudes.
O = Difference of Longitude.
C = The Angle of the Courſe.
G = 10,1015104, the Logarithmic Tangent of 51°. 38'. 9''.

Alſo, put
L = Difference of Latitudes.
D = The Diſtance ſailed.

Then we have
Caſe I. A : O : : G : *t*. C.
Caſe II. G : A : : *t*. C : O.
Caſe III. *t*. C : G : : O : A.
In Mercator's ſailing.

Alſo,
Radius : D : : *cs*. C : L.
cs. C : Radius : : L : D.
In plain ſailing.

102. In Order to the Solution of the above 3 Caſes of *Mercator*, it will be only neceſſary further to obſerve, that as in *Wright*'s Logarithms (or Meridional Parts) the Arch of the Equator equal to the Module or Radius, conſiſts of 3437,75 Minutes (54.) and theſe Minutes being *whole Numbers* (for 1 Minute is *Unit*,) expreſs the Meaſure of the *Longitude*, *Latitude*, and *Diſtance* ſailed: So in taking the Differences of the Artificial Tangents of *Briggs*'s Form, or thoſe of the common Canon, the *four* firſt Figures on the Left Hand muſt be cut off by a Comma for *Integers*; and then all Reſults will be the ſame here as in the common Ways of Solution; for 3437',75 : 4342',94 : : Tangent of 45° : Tangent of 51°. 39'. 9''.

SOLUTION *of* CASE I.

103. Suppoſe a Ship ſails from the *Lizard* in Lat. 49°. 55', North, to *Barbadoes*, in Lat. 13°. 10'. N. and the Difference of Longitude 53°. 00'. = 3180'; then the Difference of Latitude is 36°. 45'. = 2205'. The *Tangent of the Courſe* is required.

	Lat.	Comp.	Halves.	Log. Tang.
Barbadoes	13°. 10'.	—76°. 50'.	—38°. 25'.	—9,8993082
Lizard	49. 55.	—40. 05.	—20. 2½.	—9,5620477

The Difference is A = 3372,605

Then add
The Arith. Comp. of A=3372,605—6,4720346
The conſtant Log. G - - - 10,1015104
The Diff. of Long. O=3180 - 3,5024271

Sum is the required Tang. *t.* C=49°. 59′. 10″.—10.0759721

104. Then the Diſtance ſailed from the *Lizard* to *Barbadoes* is found by ſaying,

As the Co-Sine of the Courſe - 9,8081929
To Radius - - - - - 10,
So is the Difference of Latitude 2205—3,3434086
To the Diſtance ſailed - - 3429,38—3,5352157

SOLUTION *of* CASE II.

105. Suppoſe a Ship ſails from the *Lizard*, Latitude 49°. 55′. N. to *Barbadoes* in Lat. 13°. 10′. N. on a Rhumb or Courſe of 49°. 59′. 10″. N. to find the Difference of Longitude.

Add the
Arith. Computation of G - - 9,8984896
Above found Value of A=3372,605—3,5279654
Tang. of the Courſe *t.* C=49°. 59′. 10″. 10,0759721

Sum is the Difference of Long. O=3180′ 3,5024271
Or 53°. And the Diſtance ſailed is found as before.

SOLUTION *of* CASE III.

106. The Latitude you depart from, the Courſe ſteer'd, and the Difference of Longitude being given; to find the Latitude of the Port the Ship is arrived at.

Add the
Arith. Comp. of *t.* C = 49°. 59′. 10″.—9,9240279
Conſtant Logarithm G - - 10,1015104
Difference of Long. O=3180 - 3,5024271

Sum is the Value of - A=3372,605—3,5279654
Suppoſing, then, that you ſailed from the *Lizard* in Lat. 49°. 55′. North, the Complement of which is 40°. 05′, the Half of that is, 20°. 2′ ½.

Then the Logarithmic Tangent of 20°. 02′ $\frac{1}{2}$—9,5620487
Add the Value of A in Form of a Logarithm 0,3372605

The Sum is the Tangent of 38°. 25′. - 9,8993082
The Double of which is - 76°. 50′.

The Complement thereof is 13°. 10′.=the Lat. of the Port required, viz. *Barbadoes*.

107. Thus (ſays Dr. HALLEY) the dead Reckoning by the Logg, and daily Account of the Ship's Way is duly kept, and the Trouble thereof very little more than by *Plain Sailing*. Theſe are all the practical Caſes, performable by a Table of Logarithms, others fall not under our Cognizance.

108. It is here to be obſerved, that both the Complements of the Latitudes B and M (Fig. 2.) are to be eſtimated from the ſame Pole of the World, either A or D, it is not material from which; for ſuppoſe we regard the Pole D, then $\frac{90-EB}{2}=\frac{BD}{2}$; and $\frac{90+EM}{2}=\frac{MD}{2}$. But if the Pole A be referred to, then it will be $\frac{90+EB}{2}=\frac{AB}{2}$, and $\frac{90-EM}{2}=\frac{AM}{2}$. But the Ratio of the Tangents of $\frac{1}{2}$BD and $\frac{1}{2}$ MD is the very ſame as the Ratio of the Tangents of $\frac{1}{2}$ AB and $\frac{1}{2}$ AM, (See Article 77.)

109. To illuſtrate this Matter in Numbers, ſuppoſe EB=50° and EM=30°; then $\frac{1}{2}$ BD=20°, and $\frac{1}{2}$ MD =60°. Alſo $\frac{1}{2}$ AB=70°, and $\frac{1}{2}$ AM=30°.

Then the Log. Tangent of $\begin{cases} 60° — 10.2385606 — 70° — 10.4389341 \\ 20° — 9.5610659 — 30° — 9.7614394 \end{cases}$

Remainder in both equal 6774,947 6774,947

This is alſo very evident in the Line of Tangents MN (of Fig. 4.) where the Difference of the Tangents of

60° and 20°, and of 70° and 30° is the ſame, and anſwers to the two Tangents *e* 30 and *e* 50 in the SCALES of WRIGHT'S LOGARITHMS CI, and OP.

CHAP. IX.

The USE *of the* Logiſtic CURVE *and* Logarithmetic SERIES, *exemplified in* Natural PHILOSOPHY.

110. NOTHING has contributed more, perhaps, to the Advancement of *Natural* PHILOSOPHY than the Invention of Logarithms. An Inſtance of this we ſhall give in the Uſe of the *Logiſtic Curve* for duly repreſenting the *Quantity* of Air, above any given Height from the Surface of the Earth; and its Denſity at that Height; and from thence the Method of *meaſuring the Height of a Mountain, Tower,* &c. is very naturally deduced.

111. It is well known by Experiments, that the Denſity of the Air is always proportional to its Compreſſion, that is, to the Weight or Quantity of the ſuper-incumbent Atmoſphere. So that if TN (Fig. 1.) be a Part of the Earth's Surface, then the Denſities of the Air at the Earth T, and at any Heights O, S, A, above it, will be directly as the Quantity of Air, or it's Weight, at, or above the Points T, Q, S, A. But the indefinite Areas WNTY, WOQY, WPSY, WBAY, we have ſhewn (33.) are as the terminating Ordinates TN, QO, SP, AB in the *Logiſtic Curve*. Therefore it is plain, theſe Areas will repreſent the Quantities of Air above the Points T, Q, S, A; and the Denſities of the Air at thoſe Points will be as the Ordinates TN, QO, SP, AB, reſpectively.

112. Hence it follows, that if ſeveral Altitudes of Air TI, TG, TE, TC, TA, &c. be taken in *Arithmetical Pro-*

Progreſſion, then the Denſities of the Air at thoſe Heights IK, GH, EF, CD, AB, &c. will be a Series in *Geometrical Proportion*, decreaſing; and the Differences of the Altitudes IG, IE, &c. will be as the Logarithms of the Denſity at I to the Denſity at G, E, &c. ſee Chap. II.

113. Furthermore, it has been ſhewn (32.) That the *Rectangle* TNRS upon the terminating Ordinate TN, and Subtangent TS, is equal to the indefinite Area WNTY of the Curve above it; and conſequently if all the Air above the Surface of the Earth were reduced into an Atmoſphere every where *uniformly denſe* (like Water,) then all that is incumbent upon TN would be reduced into the *Rectangle* TNRS; and the Height of ſuch an uniformly denſe Atmoſphere would be the Subtangent TS.

114. And becauſe TS is the Module of the Syſtem of Logarithms TQ, TS, &c. therefore, if we find by any Means the Denſity of the Air at the Earth's Surface T, and at any Height TQ above it, we ſhall have the Logarithm TQ of the Ratio of theſe Denſities; and conſequently, can find the Height TS of the uniformly denſe Atmoſphere by this Analogy, viz. *As the Logarithm* TQ *is to the Module* TS, *ſo is the Altitude of* Q *to the Altitude of* S.

115. The Heights of the Quickſilver in the Barometer, it is well known, are the Effects of the Air's Weight or Preſſure; and are therefore always as the Denſities thereof; now it is known, by Experiments, that when the Mercury is 30 Inches high on the Surface of the Earth, it will ſink $\frac{1}{10}$ of an Inch at the Height of 97 Feet above it. Hence the Denſities will be as 30 to 29,9; therefore, the Logarithm of 30 − Log. of 29,9 = Logarithm of $\frac{30}{29{,}9}$ = 0,00145 and the Module of the Canon is 0,43429. Therefore ſay, as 0,00145 : 0,43429 : : 97 : 29053 = TS,

the Height of the uniform Atmoſphere required, which is $5\frac{1}{2}$ Miles.

116. Again, ſuppoſe on the Top of a Mountain you obſerved the Height of the Mercury to be 25 Inches, when it ſtands at 30 at the Foot of the Mountain; Query it's Height? To anſwer this, the Logarithm of $\frac{30}{25}=$ 0,079181; therefore ſay, as Logarithm of $\frac{30}{29,9}=0,00145$: Logar. of $\frac{30}{25}=0,079181$: : 97 : 5297 Feet = 1 Mile 17 Feet, the Height of the Mountain required.

117. For a *Second* EXAMPLE of the Uſefulneſs of Logarithms in *Natural* PHILOSOPHY, let it be required to find the Number of Turns (N) of the Winch of an AIR PUMP to rarify the Air R Times in a Receiver, whoſe Capacity is to that of the Barrel of the Pump as C to 1. To this End it muſt be conſidered, that the natural Rarity of the Air in the Receiver is as C, and becauſe upon the firſt Turn of the Winch it will fill the Receiver and Barrel both, the Rarity will then be as $C+1$; and therefore by one Turn it will be rarified in the Ratio of the Space C to the Space $C+1$; or of 1 to $\frac{C+1}{C}$.

118. Now this will be the conſtant Ratio of it's Rarefaction by every Turn of the Winch; therefore in 1, 2, 3, 4, 5, - - - N Turns, the reſpective Rarefactions will be the *geometrical Series* 1. $\frac{\overline{C+1}|^{1}}{C}$. $\frac{\overline{C+1}|^{2}}{C}$. $\frac{\overline{C+1}|^{3}}{C}$. - - - - $\frac{\overline{C+1}|^{N}}{C}=R$, the laſt Degree of Rarefaction of the Air; then by the *Properties* of Logarithms (9.) we have the Log. of $R=N\times$ Log. of $\frac{C+1}{C}=N\times$ Log. $\overline{C+1}-$ Log. C; and therefore $N=\frac{\text{Log. } R}{\text{Log. } \overline{C+1}-\text{Log. } C}$.

Therefore ſuppoſe R = 1000, and C = 20; then the Log. of R is 3, and Log. of $\frac{21}{20}$ = 0,021189; and ſo N = $\frac{3}{0{,}021189}$ = 141 ½, the Number of Turns for rarifying the Air 1000 Times, in a Recipient 20 Times as big as the Barrel of the Pump.

CHAP. X.

Of the NATURE, NOTATION, COMPARISON *and* ARITHMETIC *of* RATIOS.

119. FROM what has been delivered concerning the general Nature of Logarithms in the Firſt Chapter, it appears that all Numbers may be conſidered as Terms of a numerical Series in *geometrical Progreſſion*, of which Unity (1) is the firſt Term; and the Logarithms in any Syſtem are their *Exponents*, or ſhew the Places in the Series from Unity, in which they ſtand according to the Number of thoſe Terms into which the whole Ratio of 10 to 1 is divided, and which Number is called the Logarithm of 10 in each Syſtem.

120. Hence, if we ſuppoſe the ſaid Ratio of 10 to 1 to be divided into 2,3025851, or 10000000, or 7915704 equal Parts, then the Places of the 9 Digits will be indicated by their Logarithms in each reſpective Syſtem, as in the following Table; where, in the firſt Syſtem of Natural Logarithms, the Number 3 holds the 10986123d Place from Unity. In *Briggs*'s Syſtem, it is in the 4771213th Place; and in *Wright*'s Scale it ſtands in the 3776776th Place, and ſo of all the Reſt.

Numbers.	Common Logarithms.	Briggs's Logarithms.	Wright's Logarithms.
1	0.0000000	0.0000000	0.0000000
2	0.6931472	0,3010300	0,2382866
3	1.0986123	0.4771213	0.3776776
4	1.3862943	0.6020600	0.4765733
5	1.6094379	0,6989700	0.5532838
6	1.7917594	0.7781513	0.6159742
7	1.9459101	0.8450980	0.6689548
8	2.0794415	0.9030900	0.7148599
9	2.1972241	0.9542425	0.7553552
10	2,3025851	1,0000000	0,7915704

121. All that is contained in the Four Columns of this Table, is evidently the ſame as is protracted in the Four Lines VW, ST, QR, and OP; for in the firſt are contained the natural Numbers as they ſtand in the Scale of Proportionals between 1 and 10: and in the other Three are the reſpective Logarithms or Ratios of all ſuch Numbers to Unity, and to each other. Thus to the Number 3 in the Line VW, there correſponds the *natural Logarithm* 10986 in the Scale ST; *Briggs*'s Logarithm 47712 in QR; and *Wright*'s Logarithm 37768 in OP; and ſo of others.

122. Hence it appears, that the *Line of Numbers* is a LINE OF THE MEASURES of all Ratios in every Scale of Logarithms; and, in the Opinion of the great Dr. SAUNDERSON, it is the beſt Inſtrument we have for exhibiting and conveying Ideas of Ratios in all their Modifications; and when this Line is twice repeated, properly divided, and 3 or 4 Feet in Length, it will anſwer the End of a Table of Logarithms where great Exactneſs is not required.

123. Of Ratios there are two Sorts, *Logiſtical* and *Numerical*. A *Logiſtic Ratio* is that of the Diſtance of any Number from Unity (1) in the Scale of Proportionals; thus, the Ratio of 2 to 1 is the Diſtance V 2 in the *Gunter*, and the Ratio of 3 to 1 is the Diſtance V 3; and

ſo on. Alſo the Ratio of 3 to 2 is the Diſtance between V 3 and V 2, or between 3 and 2 in the *Gunter*; and thus it appears that the *Logiſtic Ratio of any one Number to another is the Difference of their Logarithms*; and is itſelf *a Logarithm.* This Ratio is thus expreſſed A : B, C : D, &c.

124. The *Numerical Ratio* is that which expreſſes the *Relation of Magnitude* which one Number has to another; and is the Quotient ariſing from a Diviſion of the two Terms or Quantities A and B, which is therefore thus expreſſed $\frac{A}{B}$ or $\frac{B}{A}$; thus ſuppoſe $A=3$, and $B=2$, then $\frac{3}{2}=1,5$ which is the Ratio of the Magnitude of 3 to 2, viz. the Ratio of $1\frac{1}{2}$ to 1. And $\frac{2}{3}=0,666$ which ſhews the Ratio of 2 to 3 is that of 1 to $\frac{6}{10}$, or 1 to $\frac{66}{100}$, or more nearly as 1 to $\frac{666}{1000}$.

125. It is evident from what has been ſaid, *that the Logiſtic Ratio* (L) *is the Logarithm of the Numerical Ratio* (N); for it has been ſhewn that $L.\frac{A}{B}=L.A-L.B=L.N.$ And thus in the *Gunter* VW, if you take the Diſtance between 3 and 2 in your Compaſſes, and placing one Foot in 1, the other will reach to 1,5. the Ratio of the Magnitude of 3 to 2.

126. Hence it is eaſy to *compare Ratios* of Magnitude with each other; thus we ſee in the Gunter, the Ratio of 3 to 2 being greater than that of 4 to 3, ſhews that 3 has greater Magnitude compared with 2, than 4 has compared with 3, that is, $\frac{3}{2}$ is greater than $\frac{4}{3}$. Thus alſo you compare ſeparate or diſcontinued Ratios, as thoſe of $\frac{7}{5}$ and $\frac{3}{2}$; for ſet one Foot of the Compaſſes in 7 and extend the other to 5, then apply that extent from 3 towards 2, and as it falls ſhort of 2, it ſhews that 7 has leſs Magnitude compared to 5, than 3 has to 2; and that extent will alſo reach from 1 to 1,4, and therefore ſhew that $\frac{3}{2}$ exceeds $\frac{7}{5}$ in the Ratio of 15 to 14.

127. On the Contrary, by inverting the Terms of the Numerical Ratios, we find the Ratio of $\frac{2}{3}$ to $\frac{5}{7}$ as 14 to 15; or that 2 has leſs Magnitude compared with 3, than 5 has compared with 7, in the Ratio of 14 to 15.

128. Hence alſo the COMPOSITION or ADDITION of *Ratios* will be very facile; for let it be required to add the Ratios of B to A and D to C; if you place them Fractionwiſe thus, $\frac{B}{A}$ and $\frac{D}{C}$, it is evident the Sum of the $L.\frac{B}{A} + L.\frac{D}{C}$, will be the $L. B - L. A + L. D - LC$, that is, the $L.\frac{BD}{AC}$, and therefore the Sum of the two Ratios of B to A and of D to C, is that of BD to AC. Thus the Sum of three Ratios B to A, D to C, F to E, will be BDF to ACE; and ſo on for a greater Number of Ratios.

129. In continued Ratios, or ſuch where the *conſequent Term* in one is the *Antecedent* of the next, as in theſe A to B, B to C, C to D, D to E, &c. the Sum of all will be the Ratio of ABCD to BCDE, or only of A to E; the intermediate Terms BCD being found on each Side of the compound Ratio, make no Alteration. Thus the Sum of the Ratios 2 to 3, 3 to 4, 4 to 5, is only that of 2 to 5, as is evident by Inſpection only of the Line VW.

130. The SUBTRACTION or RESOLUTION of Ratios is by a contrary Proceſs; thus to ſubduct the Ratio of D to C from the Ratio of B to A, we have the $L.\frac{B}{A} - L.\frac{D}{C} = L. B - L. A - L. D + L. C = L.\frac{BC}{AD}$; and therefore the Remainder is the Ratio of BC to AD. Hence it appears that to *ſubtract a Ratio* is the ſame thing as to *add it* with the *Terms inverted*.

131. The MULTIPLICATION of a *Ratio* A to B is to *add it to itſelf as often as there are Units in the Multiplier*; thus the Ratio of A to B multiplied by 2 is added twice, viz. $L.\frac{A}{B} + L.\frac{A}{B} = L.\frac{AA}{BB} = 2\,L.\frac{A}{B}$; therefore the Product is the Ratio of A^2 to B^2. Thus the ſame Ratio multiplied by 3 is that of A^3 to B^3; ſo that to multiply a Ratio is only to *involve it to a given Power*, as the *Square*, *Cube*, &c.

132. On the other Hand the DIVISION of a *Ratio* is only to extract its Square, Cube, &c. Root; thus to divide the Ratio A to B by 2, it muſt be conſidered as a *Square*, ſo that $\frac{A}{B} = \frac{aa}{bb}$; then $L.\frac{a^2}{b^2} = 2\,L.\frac{a}{b} = L.\frac{A}{B}$; which divided by 2, gives $L.\frac{a}{b} = \frac{1}{2}\,L.\frac{A}{B}$; and conſequently the *Quotient* is the Root $\frac{a}{b} = \frac{A^{\frac{1}{2}}}{B^{\frac{1}{2}}}$, or the Ratio $A^{\frac{1}{2}}$ to $B^{\frac{1}{2}}$; thus the Ratio of A to B divided by 3, or *triſected*, is that of $A^{\frac{1}{3}}$ to $B^{\frac{1}{3}}$, and ſo on for others.

133. Theſe Operations are all exemplified, as it were by Sight, in the Line V W; for double the Diſtance V 2 is = V 4, and triple the ſame = V 8, the Square and Cube of 2. Alſo $\frac{1}{2}$ V 9 = V 3, the Square Root; and $\frac{1}{3}$ V 8 = V 2, ſhewing 2 is the Cube Root of 8: and ſo on.

134. The Ratio of A to B is ſaid to be *direct*, when the Terms A and B increaſe or decreaſe together, and in an equal Degree; ſuch a Ratio may be denoted thus, $A : \frac{B}{1}$, but uſually thus, A : B.

135. But if A increaſes or decreaſes while B proportionally decreaſes or increaſes, then the Ratio of A to B is ſaid to be *inverſe*, that is, A is inverſely as B, and is thus denoted $A : \frac{1}{B}$, and often $A : B^{-1}$.

136. When any Quantity A is inverſely as the ſquare of another B, then it is thus expreſſed $A : \frac{1}{B^2}$, but more commonly and elegantly now, $A : B^{-2}$; for ſince the *direct Ratio* of A to BB is $A : B^2$ or $A : B^{+2}$, ſo the *inverſe Ratio* being juſt the contrary, muſt have a contrary or negative Index of the Power 2.

137. For the ſame Reaſon if A be as the Square Root of B, it is thus expreſſed $A : \sqrt[2]{B}$, or now more commodiouſly in this Form $A : B^{\frac{1}{2}}$. But if the Ratio of A be inverſely as the Square Root of B, that is, if $A : \frac{1}{\sqrt[2]{B}}$, then it is wrote $A : B^{-\frac{1}{2}}$.

138. When A^2 is directly as B^3, then $A : B^{\frac{3}{2}}$, or A is ſaid to be in a *ſeſquiplicate Ratio* to B; and this Ratio was formerly wrote thus, $A : B\sqrt{B}$.

139. In the Compariſon of Ratios conſiſting of the Powers of Quantities, 'tis cuſtomary to denote ſuch Powers by Symbols, or Letters, as A^m, B^n, C^r, &c. Then if the Ratio be $A^n : B^m$; it will be $A : B^{\frac{m}{n}}$. Again, if $A : \frac{1}{B^m}$; then $A : B^{-m}$, or $A : B^{0-m}$. Alſo if $A^n : \frac{1}{B^m}$, then we have $A : B^{\frac{0-m}{n}}$, becauſe $B^0 = 1$, or $\frac{1}{B^m} = \frac{B^0}{B^m}$. Hence, if $A^n : \frac{B}{B^m}$ then we have $A : B^{\frac{1-m}{n}}$. And if $A^n : \frac{B^r}{B^m}$, then $A : B^{\frac{r-m}{n}}$. Suppoſe $A^n : \frac{B^r B^s}{B^m}$, then $A : B^{\frac{r+s-m}{n}}$. And laſtly, let $A^{\frac{n}{p}} : \frac{B^r}{B^m}$; then $A : B^{\frac{rp-pm}{n}}$; and thus you proceed in forming compound

Indices in all other Caſes; to know which will be extremely neceſſary for the Student in the *Modern* PHILOSOPHY and MATHESIS.

CHAP. X.

The PRINCIPLES *of the* Coteſian GEOMETRY *explained; and applied to the finding of* FLUENTS *by the* MEASURES *of* RATIOS *and* ANGLES.

140. FROM all that has hitherto been ſaid, it muſt abundantly appear, that every thing which concerns the Meaſures of Ratios muſt have it's Foundation in the Doctrine of LOGARITHMS; and conſequently as Mr. Cotes in his *Harmonia Menſurarum* has invented a new and elegant Method of finding the Fluents of given Fluxions, by reducing them to Ratios, and taking their Meaſures from a Table of Logarithms, or the trigonometrical Canon for circular Arches, it may be proper here to explain the Principles on which his Method is founded.

141. For this Purpoſe his Notation of a Ratio is ſomewhat different from the common Form. In the fluxionary Equation $r \times \frac{\dot{y}}{y} = \dot{x}$ (49) he neglects the Meaſure $\dot{x}$, and conſiders only it's equal $r \times \frac{\dot{y}}{y}$, which he transforms into this Notation $r\,\Big|\frac{\dot{y}}{y}$, and calls it the Meaſure of the Ratio of $\dot{y}$ to y in the Scale of Logarithms whoſe Module is the Radius (r) of the Circle. Accordingly the Ratio itſelf is $r\,\Big|\frac{CN}{CL}$; and the Meaſure of that Ratio in the Scale for the Module (r), is the Fluent of the Fluxion $\frac{\dot{y}}{y}$.

142. Mr. COTES's general *Formula* is $R\left|\frac{R+T}{S}\right.$; the Quantities R, S, T, denoting the *Ratio* or *Angle*, by the Meaſure of which the Fluent of a Fluxion is to be exhibited. When R is the Square Root of an affirmative Quantity, then they expreſs a Ratio, viz. the Ratio of $R+T$ to S; but if R be the Square Root of a negative Quantity, they denote an *Angle*, which Angle is always that whoſe *Tangent* and *Secant* are to the *Radius* as T and S to R; the negative Quantity being converted into a poſitive one by changing the Sign.

143. Whence it is manifeſt that a *Fluent has two Values*, viz. one *Logometrical*, conſiſting in the *Meaſure of a Ratio*; and the other *Trigonometrical*, being the Arch of a Circle of the modular Radius. To make this Matter of a *two-fold* Expreſſion of the Fluent evident, let us conſider the Ratio of $\frac{T+R}{T-R}$ *ad Modulum* 1; this Ratio is compounded of the two ſimple Ratios $T+R$ to 1, and 1 to $T-R$; the Fluxion of the firſt is $\frac{\dot{T}}{T+R}$, and of the laſt it is $\frac{-\dot{T}}{T-R}$ (ſee 20, 23.) and the Sum of both theſe together is $\frac{-2R\dot{T}}{T^2-R^2}$, which is the Fluxion of the Ratio $\frac{T+R}{T-R}$ to the Module 1.

144. But to bring this Fluxion into a proper Form, ſuppoſe the Module encreaſed to $\frac{1}{2}R$; then $\frac{1}{2}R \times \frac{-2R\dot{T}}{T^2-R^2} = \frac{-R^2\dot{T}}{T^2-R^2}$, which is now the Fluxion of the ſame Ratio $\frac{T+R}{T-R}$ to the Module $\frac{1}{2}R$.

145. And becauſe it is $T+R : S :: S : T-R$, therefore $T^2-R^2=S^2$; and ſo $\frac{-R^2\dot{T}}{T^2-R^2}=\frac{-R^2\dot{T}}{S^2}$, which is the Fluxion of the Ratio in the neceſſary Form.

146. Since a mean proportional biſects the Ratio of the two Extremes, therefore, $\frac{T+R}{S}$ will be half the Ratio of $\frac{T+R}{T-R}$; and this will be the Fluent of the Fluxion $\frac{-R^2\dot{T}}{S^2}$ in a Scale of Logarithms whoſe Module is double the former; that is, the true Fluent will be (in the *Coteſian* Style) thus expreſſed $-R\,\Big|\frac{T+R}{S}$.

147. We ſhall illuſtrate this *Logometric Doctrine* by Lines in Fig. 1. It has been obſerved (21.) that there is no definite Meaſure of any Ratio but in a Scale or Syſtem of Logarithms whoſe Module is given. And every Ratio $\frac{A}{B}$, where no Module is ſpecified, is ſuppoſed to belong to the *natural Syſtem* whoſe Module is 1, or where $t=1$ $=TS$ the Subtangent (24, &c.) Hence $1\,\Big|\frac{T+R}{T-R}$ is the Meaſure of the Ratio $T+R$ to $T-R$ in the natural Scale.

148. Put $LM=T+R$, $QO=T-R$, and $TN=S$; then becauſe $QO : TN :: TN : LM$; the Line QL, (the Ratio of LM to QO or $T+R$ to $T-R$) will be double the Line TL, which is the Ratio of $T+R$ to S. We have ſeen the Fluxion of QL in the natural Scale is $\frac{-2R\dot{T}}{S^2}$ (143). And in the Syſtem where the Module $TS=\frac{1}{2}R$, that Fluxion is $\frac{-R^2\dot{T}}{S^2}$ (145); and conſequently the Flent or Logarithm QL will be varied in the Ratio of the Modules, viz. of 1 to $\frac{1}{2}R$.

149. But if the Fluent of that Fluxion muſt be the

Meaſure of but half that Ratio QL, that is, if TL muſt equal QL, or be doubled, then ſo muſt the Module TS $(=\frac{1}{2}R)$ likewiſe; therefore the Fluent TL will be the Meaſure of the Ratio $\frac{T+R}{S}$ to the Module R; ſo that at laſt we have found that the Fluxion of the Ratio $-R\left|\frac{T+R}{S}\right.$ is $\frac{-R^2\dot{T}}{S^2}$.

150. We are next to ſhew, that this very Fluxion (in the poſitive Form $\frac{R^2\dot{T}}{S^2}$) has alſo for its Fluent the Arch of a Circle, whoſe Radius is the Module R; for (in Fig. 2.) let C*a* be the Arch of a Circle or Meaſure of the Angle C A *a* whoſe Radius, Tangent and Secant are AC, CN, AN denoted by R, T, and S, reſpectively. Then it has been ſhewn (66) that FN $(\dot{T}) : ab :: AN^2 : AC^2 :: S^2 : R^2$; conſequently $ab = \frac{R^2\dot{T}}{SS} =$ the Fluxion of the Arch C *a*, the ſame as before.

151. Hence it is evident, that every Fluxion of this Form, and all others which depend upon it, will have a Fluent of two different Values, viz. one *Logometrical*, conſiſting in the Meaſure of a Ratio; and the other *Trigonometrical*; in the firſt, the Symbols R, T, and S, are only the Terms of a Ratio; in the latter they expreſs the Relation of the *Radius*, *Tangent*, and *Secant* of an Angle only, whoſe Meaſure is the Fluent ſought, when adapted to the proper Module.

152. From what has been ſaid, a *Formula* for Practice may be eaſily deduced, both for the logometrical and trigonometrical Meaſure of a given Ratio $\frac{A}{B}$. For with reſpect to the firſt, it is plain, that the original Meaſure is a Logarithm L in the Natural Scale whoſe Module is 1, and is thus expreſſed $1\left|\frac{A}{B}\right. = L$.

153. But as we have no Tables of Logarithms for that Syſtem, and the Value of L without Logarithms can only be inveſtigated in an infinite Series (24.) therefore, it muſt be ſought in that Syſtem, where Logarithms are calculated for all Numbers, viz. the Canon in common uſe, whoſe Module is 043429, &c. $=m$ (41.) then, ſince Logarithms are as the Module of the Syſtems, we have $1 : m :: L : mL=l$, the Logarithm of $\frac{A}{B}$ in *Briggs's* Scale, or $m\,\Big|\,\frac{A}{B}=l$. Therefore $L=\frac{1}{m}l=2.302585\,l$, or making $N=2,302585$, &c. (41.) we have $L=N\,l$.

154. But if the Value L of the Meaſure of the Ratio A to B be expreſſed in any other Syſtem whoſe Module is given, as R; then $1 : R :: N\,l : R\,N\,l$, = Meaſure of the Ratio in that Syſtem; and conſequently, $R\,\Big|\,\frac{T+R}{S}=R\times N\,l$, the *Formula* required, conſiſting of known Quantities R, N, l.

155. It is to be obſerved, that in *Logometric Ratios*, the Letters R, S, T, denote only the Parts of a *Plain Triangle*, or Quantities proportional to them; of which R and S are the Legs or Sides, and T is the Hypothenuſe; thus, ſuppoſe $R=3$, $S=4$, and $T=5$; then, $T+S=9$, $T-S=1$, and ſo $\frac{T+R}{T-R}$ is the Ratio of 8 to 2, or 4 to 1; the Half of which is the Ratio of 2 to 1, or $R\,\Big|\,\frac{T+R}{S}=R\,N\,l$. (See 149.)

156. While R is a *poſſible* Quantity, the Fluent will be the *Meaſure of a Ratio*; but when $R=\sqrt{-1}$, or $\sqrt{-e}$, or $\sqrt{\frac{-e}{f}}$, or any other *impoſſible* Expreſſion,

it's Sign muſt be changed, and then it will be poſſible, and the Fluent will be the *Meaſure of an Angle* to a given Radius. Thus, if $\pm R=\sqrt{-e}$ by changing the Signs on both Sides we have $\mp R=\sqrt{+e}$, and therefore $R^2=e$. Therefore the Logometric Expreſſion $T^2-R^2=S^2$ becomes (in that Caſe) $T^2+R^2=S^2$. And the Fluent will now be $R\left|\frac{T+R}{S}\right.$, the Meaſure of an Angle, given by the *Radius* R, *Tangent* T, and *Secant* S.

157. Mr. Cotes uſed ſometimes to expreſs this trigonometric Fluent thus, $R\left(\frac{R.T}{S}\right.$; to diſtinguiſh it from the other. As there was a *Ratio Modularis* (30) in *Logiſtical Meaſures*, ſo there is an *Angulus Modularis* in the *trigonometric* ones. And this *Modular Angle* is that whoſe Meaſure, or Arch A, is equal to Radius, and is thus found; as the Circumference of a Circle 3,14159, &c. is to the Diameter 1, ſo is 180 Degrees to $57^{\circ}.\ 17'.\ 44''=A$, the *Modular Angle* in circular Meaſures.

158. As the Meaſure of an Angle is always proportionate to the Radius, the Radius will be the *Module* of the *Trigonometrical Canon*, and putting the Radius $=1$, we find the Length of any other Arch G by this Analogy; as 57,2957795, &c. $(=A)$ is to Radius$=1$, ſo is the given Arch G, to $\frac{G}{A}=$ the Length or Meaſure of the Arch G in decimal Parts of the Radius. Put $\frac{1}{A}=$ 0.01745329, &c.$=n$, then the *trigonometrical Meaſure* of any Angle G will be nG, when $R=1$; but in all other Caſes, that Meaſure will be RnG.

159. Having thus inveſtigated the Meaſures of Ratios and Angles, we ſhall next ſhew how Fluents of Fluxions are thereby found, and many very uſeful Propoſitions in

Geometry receive the moſt ſimple and elegant Solutions from thence. Let it be required, for Example, to *ſquare the Circle*: in order to this, let A Ca be the Sector (Fig. 2.), whoſe Arch aC$=30^\circ=$G. The Fluxion of this Arch is $\frac{R^2\dot{T}}{S^2}=ab$, (150.) this multiplied by $\frac{1}{2}$ R is $\frac{1}{2}R\times\frac{R^2\dot{T}}{S^2}=$ to the ſmall Sector Aab, which is the Fluxion of the Sector A Ca. But the Fluent of $\frac{R^2\dot{T}}{S^2}$ is the Meaſure of the Arch C$a=$RnG; therefore $\frac{1}{2}R^2 n$G will be the Area of the Sector of 30° to the Radius R, as a Module.

160. If R$=1$, then RnG$=n$G$=0.01745329\times30=0,523598$, &c. and ſo $\frac{1}{2}n$G$=0,261799$ the Area of the Sector required. Then becauſe $12\times30=360$, we have $0,261799\times12=3,1415926$, &c. the Area of a Circle, whoſe Radius $=1$, or Diameter $=2$.

161. Mr. COTES has given us a Table of general Forms of Fluxions and their Fluents, with which if other analogous Fluxions are compared, their Fluents may be readily found. His firſt *Formula* is $\frac{d\dot{z}z^{\theta n-1}}{e+fz^n}$, the Fluent of which (when $\theta=1$), is $\frac{d}{nf}\Big|\frac{e+fz^n}{e}$. The Truth of which will thus appear; make $e+fz^n=x$, then you have the Fluxions on each Side $nf\dot{z}z^{n-1}=\dot{x}$ and $\frac{nf\dot{z}z^{n-1}}{e+fz^n}=\frac{\dot{x}}{x}$, and therefore $\frac{d\dot{z}z^{n-1}}{e+fz^n}=\frac{d}{nf}\times\frac{\dot{x}}{x}$, whoſe Fluent is the Meaſure of the Ratio of x to e, or of $+fz^n$ to e, *ad Modulum* $\frac{d}{nf}$, as above expreſſed.

162. The ſecond general Form is $\frac{d\dot{z}z^{\theta n+\frac{1}{2}n-1}}{e+fz^n}$; and its Fluent (when $\theta=0$) is $\frac{2}{ne}$ R$\Big|\frac{R+T}{S}$; this will appear

by putting $\frac{e}{f} = -R^2$, $Z^n = T^2$, and $\frac{e}{f} + z^n = T^2 - R^2 = S^2$; for then we ſhall have $Z^{\frac{1}{2}n} = T$, and $\frac{1}{2}n\dot{z}z^{\frac{1}{2}n-1} = \dot{T}$; multiply one Side into $\frac{e}{e+fz^n}$, and the other into its equal $\frac{-R^2}{S^2}$, and you will have $\frac{\frac{1}{2}ne\dot{z}z^{\frac{1}{2}n-1}}{e+fz^n} = \frac{-R^2\dot{T}}{S^2}$; laſtly, multiply both Sides by $\frac{2d}{ne}$, and you have $\frac{d\dot{z}z^{\frac{1}{2}n-1}}{e+fz^n} = \frac{2}{ne} \times -\frac{dR^2\dot{T}}{S^2}$, whoſe Fluent $\frac{2dR}{ne}$ ($\frac{R.T}{S}$ is the Fluent ſought (by Art. 146, &c.)

163. When $\frac{e}{f} = -R^2$ is impoſſible, then $\frac{e}{f} = R^2$ will be poſſible, and the Fluent will then be the Meaſure of an Angle G, whoſe *Radius*, *Tangent*, *and Secant*, are as R, T, and S, reſpectively, but whoſe Module (or true Radius) is $\frac{2}{ne}dR$; and therefore its Meaſure, or the real Fluent will be $\frac{2dR}{ne} \times nG$. But when $f = 1$, then $d = R^2 = e$, and ſince $n = 2$, the Fluent is then barely R*n*G (158.)

164. Mr. COTES has given Tables of 18 of theſe fluxionary *Formulas*, with the Method of continuing them *ad infinitum*. A Specimen of the Manner in which theſe Forms are applied to uſe we ſhall give in the Problem of *ſquaring the Aſymptotic Space* A B E F *(Fig. 6.) of the Rectangular Hyperbola* G B H, whoſe Power A C D B $= 1$, the Square of $CA = 1 = a$; let $AF = x$, $EF = y$, and $Ff = \dot{x}$. Then, from the Nature of the Curve, it is $a : a + x :: y : a$, and ſo $y = \frac{aa}{a+x}$; and therefore the Fluxion of the Area is, $y\dot{x} = \frac{aa\dot{x}}{a+x}$.

165. Now this Fluxion is ſimilar to the *firſt Form* of the Tables, viz. $\frac{d\dot{z}z^{\theta n-1}}{e+fz^{n}}$, as will appear by putting $d=aa$, $z=x$, $\theta=1$, $n=1$, $e=a$, $f=1$, for then it becomes the ſame. In the firſt Table, therefore, againſt $\theta=1$, you have the Fluent $\frac{d}{nf}\left|\frac{e+fz^{n}}{e}\right.$, that is, $aa\left|\frac{a+x}{a}\right.$ $=AC^{2}\left|\frac{AC+AF}{AC}\right.=1\left|\frac{1+x}{1}\right.=Nl$ (153) = the Space or Area ABEF, as required.

166. From hence it is evident, that the ſaid Space is the natural Logarithm of $1+x$ to the Modulus ACDB $=1$; whence if $x=1$, then $CF=1+x=2$; and the hyperbolic Space ABEF is $Nl=0{,}6931472$ (ſee Art. 120) Again, let $AL=2$ or $CL=3$, then the Area ABML will expreſs the Ratio of CL to AC, or of 3 to 1; and will therefore be the natural Logarithm of 3, viz. 1,0986123. And it is on this Account that theſe natural Logarithms have been commonly called *Hyperbolical Logarithms*.

167. But the Logarithms of any other Syſtem are as much *hyperbolical* as theſe; for if the Power ACDB of the *Hyperbola* be a Square or Rectangle = 0,434294, &c. then the Areas ABEF, and ABML will be the *Briggian* Logarithms of 2 and 3 in the common Canon; and they will be the Logarithms of the ſame Numbers in *Wright*'s Syſtem, if the Module ACDB be = 0.343775, &c.

168. I ſhall conclude with one Example more. Mr. COTES has ſhewn (*Harmonia Menſ.* Page 79.) that the Fluxion of a FIGURE of SECANTS is $\frac{rr\dot{y}}{\sqrt{yy-rr}}$, which is analogous to the Sixth *Formula* of his Tables, viz. $\frac{d\dot{z}z^{\theta n+\frac{1}{2}-1}}{\sqrt{e+fz^{n}}}$; becauſe if for z, n, θ, d, e, f, be written y, 2, 0, rr, $-rr$, 1, it will then be reduced to

the given Fluxion; the Fluent of which (againſt $\theta=0$) is $\frac{nf}{2}dR\left|\frac{R+T}{S}\right.$.

169. In this 6th Table, the Values of R, T, S, are ſuch as make them in Proportion to each other, as $\sqrt{yy-rr}$, n, r; or as the *Tangent* ET, *Secant* CT and *Radius* CE, of the Angle ECT (in Fig. 2.) The above Fluent therefore will be $CE^2\left|\frac{CT+ET}{CE}\right.$. But this Area or Sum of all the Secants on the Arch EB is to the Sum of ſo many Radii on the ſame Arch, *as the nautical Magnitude of the Arch is to its true Magnitude*; that is, as $CE^2\,\frac{CT+ET}{CE}$ to $CE\times EB$, or as $CE\left|\frac{CT+ET}{CE}\right.$ to EB.

170. Hence it appears that the nautical Magnitude of the Arch EB is $CE\left|\frac{CT+ET}{CE}\right.$, or $CE\left|\frac{CE}{CT-ET}\right.$ (by 145) and therefore *is the Meaſure of the Ratio of the Radius* CE *to the Tangent of Half the Co-latitude* CT−ET (=KD) *to the Radius* CE, as a MODULE; the ſame Thing that was otherwiſe ſhewn in Chap. IV.

FINIS.

ERRATA.

Page 14. Art. 49. in the 2d, 4th, and 5th Line, for OQ, read PQ.

In the Mariner's Mirror, the Reader is deſired to make the following Corrections, *viz.* Part II. Preface, page 2. line 22. *dele* the ſtar.—Page 11. laſt line but one, for 11° 37′ read 19′ 30′.—Page 15. line 13, for *Diameter*, read *Semidiameter.* ibid, line laſt but one, for *formed*, read *found.*—Page 19, line 13, for *Spheriod*, read *Spheroid.*—Page 20, line 2, for y, read s. line 3, 8, and 10, for xy, read xs. line 11, for As, read as. Page 30. line 1, for 30^h, read 30°. ibid, for 28^h, read 28°. line 12. for 60^h, read 60°.

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